Maximin and miniMax Regret

1. The Maximin principle
2. Normalisation
3. Indifference; equal preference
4. Graphing decision problems
5. Dominance
The *Maximin* principle

**Maximin and miniMax Regret**

1. **The *Maximin* principle**

2. Normalisation

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**Definition (The *Maximin* principle)**

Assume only minimally preferred outcomes occur and choose actions that lead to most preferred among these.

- *Maximin* and *miniMax Regret* follow *Maximin* principle: original values vs regrets
- *Maximin* principle is main decision principle used under complete uncertainty
- We’ve seen *Maximin* and *miniMax Regret* on decision tables, but what about more complex decision problems (e.g., multiple decision points)?
Multi-stage decisions

Example (Product development)
You head the R&D department of a small manufacturing company. Your company is working on a new product. The company must decide whether to develop a prototype, and, if this is successful, the scale of production (i.e., the size of the factory).

Questions
- What does Maximin or miniMax Regret mean in this problem?
- Is there a decision-table representation?
The Maximin principle

Node evaluation

- What does Maximin mean in a tree?
  - Maximin eliminates branches in chance nodes (i.e., prunes the tree)
  - Reduces problem to that of certainty

Each decision problem is assigned a ‘value’ by a decision rule

The Maximin algorithm for decision trees:

1. Begin with the leaves of the tree
2. At each parent:
   1. if a chance node, Maximin prunes all children except the minimally preferred
   2. if a decision node, the elimination principle, eliminates all children except the maximally preferred
   3. propagate the (unique) value up to the parent node
3. Repeat the previous step until the root is reached

- Value of root the value of the problem (under Maximin); i.e., value which Maximin assigns to the problem
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Problem representation: decision tables

- Observation: each action-state pair uniquely determines an outcome
- Model as a 2-ary (dyadic) function: \( \omega : \mathcal{A} \times \mathcal{S} \to \Omega \)

Represented as a table:

\[
\begin{array}{ccc}
\omega & o & \bar{o} \\
\mathcal{A} & F & A & A \\
& D & B & C \\
\end{array}
\]

Decision tables:
- row = action column = state
- Interpretation: \( B = \omega(D, o) \) means “B is the outcome of action D in state o”;

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Trees and tables

- Multiple trees may correspond to the same table
- Going from tables (normal form) to trees (extensive form) is straightforward, but the converse can be tricky
- Which representation is better: trees or tables?
- Which representation facilitates decision analysis most?

Multi-stage decisions

- Company uses factory to full capacity, making large profits. Demand satisfied
- Company only uses part of factory's capacity, breaks even. Demand satisfied
- Company uses factory to full capacity; makes moderate profits. But unsatisfied demand causes customer dissatisfaction
- Company uses factory to full capacity; makes moderate profits. Demand satisfied
- Company loses development costs
- No gains, no loss but no product
Multi-stage decisions

Example (Product development)
You head the R&D department of a small manufacturing company. Your company is working on a new product. The company must decide whether to develop a prototype, and, if this is successful, the scale of production (i.e., the size of the factory).

Questions
- What does Maximin or miniMax Regret mean in this problem?
- Is there a decision-table representation?

Actions to strategies

In a decision tree:
- Recall that a decision table is a representation of the outcome mapping \( \omega : A \times S \rightarrow \Omega \)
- Observation: following a path from the root to a leaf leads to a unique outcome
- Generalising:
  - A ‘state’ must include branches at chance nodes
  - An ‘action’ must include branches at decision nodes

Definition (Strategy)
A strategy (or policy or plan) is a procedure that specifies the selection of an action at every reachable decision point.
A strategy must specify an action at each reachable decision point; e.g., “Authorise development (Au), if development succeeds (s), then build large factory (L)”

- States: $s_1$, $s_2$, $s_3$  
  $s, h$, $s, l$, $f$

**Normalisation**

**Normalisation**

- Encoding:
  - $\alpha/A$ says: At the decision node reached via path $\alpha$ choose action $A$.
  - Example: $Au; s/S$:
    If development has been authorised (Au) and has succeeded (s), choose to build a small factory (S).

- Strategies for this problem:
  - $A_1$  $Au; s/L$
  - $A_2$  $Au; s/S$
  - $A_3$  $Ab$
Normalisation

**Outcome values:**

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>fc,lp,sat</td>
<td>10</td>
</tr>
<tr>
<td>pc,be,sat</td>
<td>4</td>
</tr>
<tr>
<td>ldc</td>
<td>-1</td>
</tr>
<tr>
<td>fc,mp,dis</td>
<td>5</td>
</tr>
<tr>
<td>fc,mp,sat</td>
<td>8</td>
</tr>
<tr>
<td>sq</td>
<td>0</td>
</tr>
</tbody>
</table>

**Decision table:**

<table>
<thead>
<tr>
<th>$s, h$</th>
<th>$s, l$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Au;s/L</td>
<td>fc,lp,sat</td>
<td>pc,be,sat</td>
</tr>
<tr>
<td>Au;s/S</td>
<td>fc,mp,dis</td>
<td>fc,mp,sat</td>
</tr>
<tr>
<td>Ab</td>
<td>sq</td>
<td>sq</td>
</tr>
</tbody>
</table>

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Exercises

- Find the Maximin and miniMax Regret strategies for this problem.
- Evaluate this problem under MaxiMax, Maximin, miniMax Regret using both normal and extensive forms.
Representing information

Consider the fund-raiser example.

- Decision before weather known:
  - Decision nodes part of the same information set
  - Possible strategies: F, S only

- Decision after weather known:
  - Decision nodes distinguishable
  - Possible strategies: viz. d/F, d/S, w/F, w/S

Maximin and miniMax Regret

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Indifference: equal preference

- Which action below is preferred above under *Maximin*?

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Definition (Indifference)

If two actions $A$ and $B$ are *equally preferred* then the agent is said to be *indifferent* between $A$ and $B.

- Indifference means an agent prefers two alternatives equally, not that it doesn’t *know* which it prefers.

Indifference classes

Definition (Indifference class)

An *indifference class* is a non-empty set of all actions/outcomes between which an agent is indifferent.

- For a given action $A \in \mathcal{A}$, the indifference class of $A$ is given by

$$I(A) = \{a \in \mathcal{A} \mid V(a) = V(A)\}$$

- Indifference classes partition set of all actions
- Different agents have different preferences over outcomes/actions, hence different indifference classes
- Different decision rules evaluate actions differently; *i.e.*, produce different indifference classes
1. The Maximin principle

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**Graphical representation**

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>D</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>E</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

Let $v_i(a) = v(a, s_i)$ be the value of action $a$ in state $s_i$. Each action $a$ corresponds to a point $(v_1, v_2)$, where $v_i = v(a, s_i)$. 

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Indifference curves: *Maximin*

For the pure actions below:

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>D</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>E</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

Consider curves of all points representing strategies with same *Maximin* value; i.e., *Maximin* indifference curves.

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Graphing regret

- Consider three actions:

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>5</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

- Regrets and indifference curves for *miniMax Regret* in blue

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**Exercises**

In regard to preference over actions, what is the relation between *Maximin* and *miniMax Regret*?
**Maximin and miniMax Regret**

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**River example**

Example (River logistics)

Alice’s warehouse is located at X on a river that flows down-stream from C to A. She delivers goods to a client at C via motor boats. On some days a (free) goods ferry travels up the river, stopping at A then B and C, but not at X.

The fuel required (litres) to reach C from each starting point:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>X</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>To C</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Alice wants to minimise fuel consumption (in litres).
River example

Alice considers three possible ways to get to C (from starting point X):

A: via A, by floating down the river
B: via B, by travelling up-stream to B
C: by travelling all the way to C

Outcomes are measured in litres left in a four-litre tank.

Exercise

Let $w: \Omega \rightarrow \mathbb{R}$ denote fuel consumption in litres. What transformation $f: \mathbb{R} \rightarrow \mathbb{R}$ is responsible for the values $v: \Omega \rightarrow \mathbb{R}$ in the decision table?

- Axes correspond to payoffs in each of the two states; i.e., payoff $v_1$ in state $s_1 = f$ and $v_2$ in $s_2 = \bar{f}$
- Actions graphed below:

<table>
<thead>
<tr>
<th></th>
<th>$f$</th>
<th>$\bar{f}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

- Option C not a better response than B under any circumstances (i.e., in any state)
- C worse than B in some cases and no better in all others; C can be discarded
Generalised dominance

**Definition (Strict dominance)**
Strategy \( A \) *strictly dominates* \( B \) iff every outcome of \( A \) is more preferred than the corresponding outcome of \( B \).

**Definition (Weak dominance)**
Strategy \( A \) *weakly dominates* \( B \) iff every outcome of \( A \) is no less preferred than the corresponding outcome of \( B \), and some outcome is more preferred.

<table>
<thead>
<tr>
<th></th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>3</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>( B )</td>
<td>4</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>( C )</td>
<td>5</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

**Exercise**
Which strategies in the decision table shown are dominated?

Dominance and best response

**Corollary**
*Strategy \( A \) strictly dominates \( B \) iff \( A \) is a better response than \( B \) in each possible state.*

**Corollary**
*Strategy \( A \) weakly dominates \( B \) iff \( A \) is a better response than \( B \) in some possible state and \( B \) is not a better response than \( A \) in any state.*

**Dominance principle**
A rational agent should never choose a dominated strategy.
Admissible actions

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
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<td>2</td>
<td>3</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

A decision rule is said to satisfy (strict/weak) dominance elimination if it never chooses actions that are (strictly/weakly) dominated.

- Dominated actions can be discarded under any rule that satisfies dominance elimination.
Dominance summary

Rules that satisfy strict/weak dominance elimination.

<table>
<thead>
<tr>
<th>Rule</th>
<th>Strict</th>
<th>Weak</th>
</tr>
</thead>
<tbody>
<tr>
<td>MaxiMax</td>
<td>√</td>
<td>×</td>
</tr>
<tr>
<td>Maximin</td>
<td>√</td>
<td>×</td>
</tr>
<tr>
<td>Hurwicz’s</td>
<td>√</td>
<td>×</td>
</tr>
<tr>
<td>miniMax Regret</td>
<td>√</td>
<td>×</td>
</tr>
<tr>
<td>Laplace’s</td>
<td>√</td>
<td>√</td>
</tr>
</tbody>
</table>

Exercise

Verify the properties above.

Rule axioms

The following criteria can be used to assess the suitability of decision rules:

Axiom of dominance

A decision rule should never choose a dominated action.

Axiom of invariance

A decision rule’s choices should be independent of representation.

Axiom of solubility

A decision rule should always select at least one action.

Axiom of independence

Adding a duplicate state should not affect a rule’s decision.
Summary: decisions under complete uncertainty

- *Maximin* in extensive form
- Multi-stage decisions
- Extensive to normal form translation
- Information in extensive form
- Graphical visualisation
- Indifference
- Dominance and admissibility