8. Inclusion-Exclusion

COMP6741: Parameterized and Exact Computation

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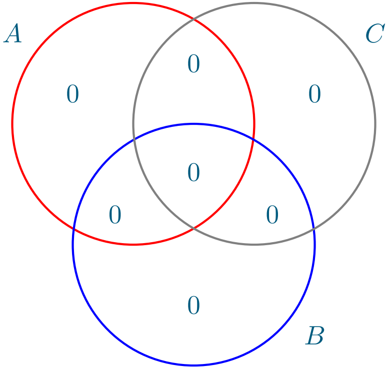
Outline

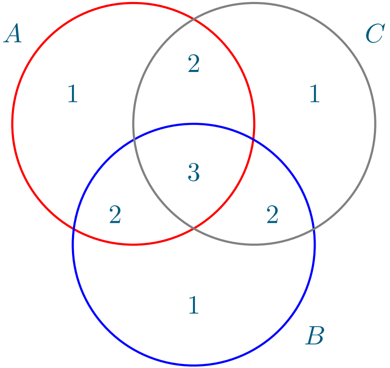
Outline

... for 3 sets

$|A \cup B \cup C| =$

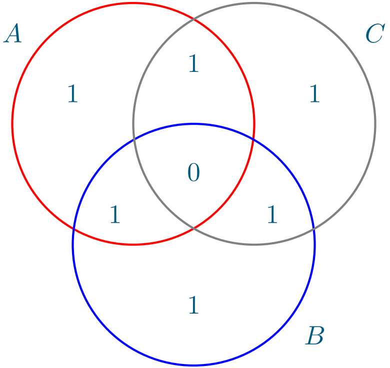


$|A \cup B \cup C| = |A| + |B| + |C|$



... for 3 sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$$



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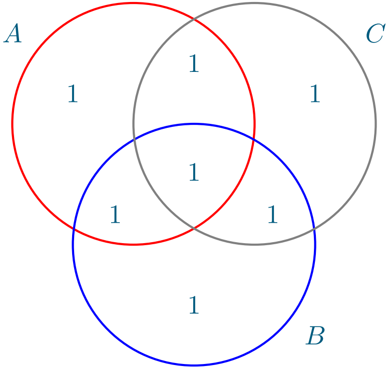
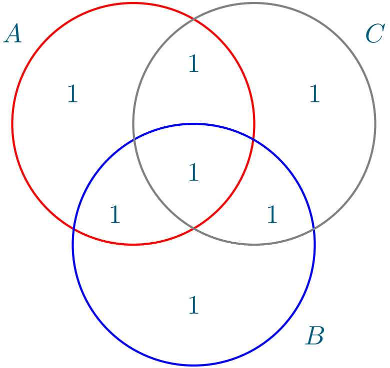


… for 3 sets

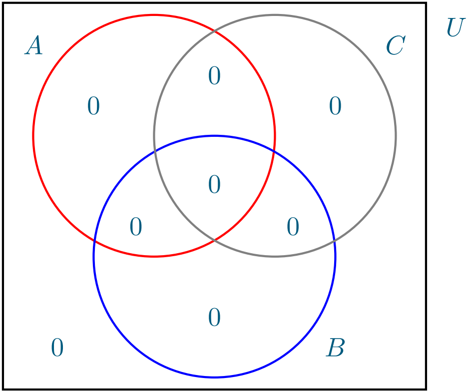
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$|A \cup B \cup C| = \sum_{X \subseteq \{A,B,C\}} (-1)^{|X|+1} \cdot \left|\bigcap X\right|$$



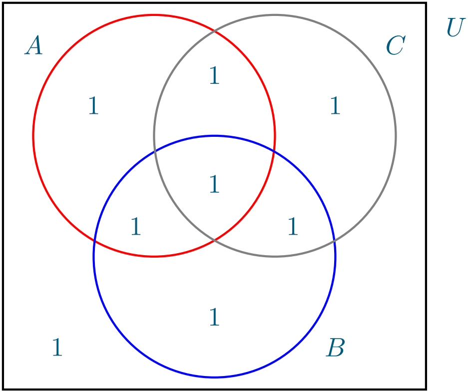
... intersection version

$|A \cap B \cap C| =$



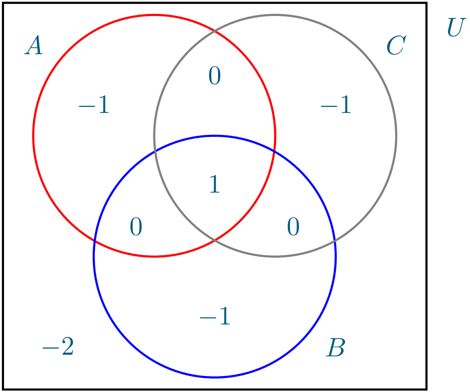
... intersection version

$|A \cap B \cap C| = |U|$



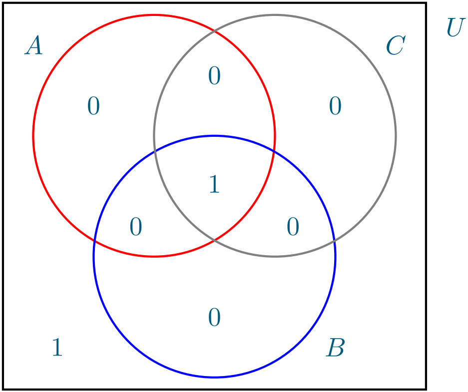
... intersection version

$|A \cap B \cap C| = |U| - |\overline{A}| - |\overline{B}| - |\overline{C}|$



... intersection version

$$|A \cap B \cap C| = |U| - |\overline{A}| - |\overline{B}| - |\overline{C}| + |\overline{A} \cap \overline{B}| + |\overline{A} \cap \overline{C}| + |\overline{B} \cap \overline{C}|$$



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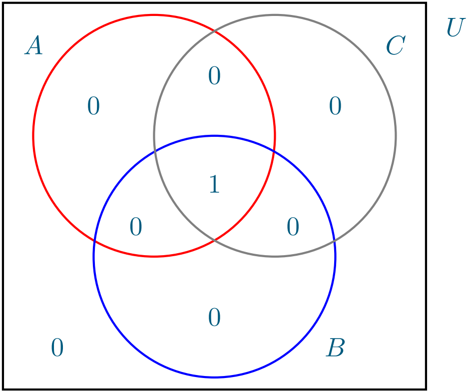

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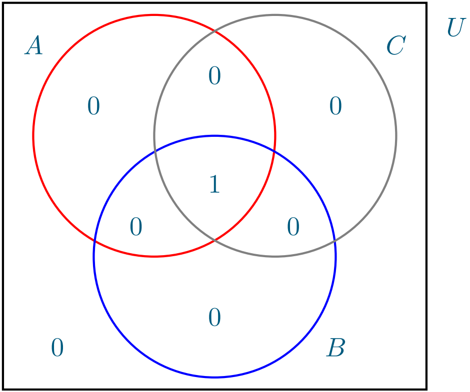
$$|A \cap B \cap C| = \sum_{X \subseteq \{A,B,C\}} (-1)^{|X|} \cdot \left| \bigcap \overline{X} \right|$$



Inclusion-Exclusion Principle – intersection version

Theorem 1 (IE-theorem – intersection version)

Let $U = A_0$ be a finite set, and let $A_1, \ldots, A_k \subseteq U$.

$$\left| \bigcap_{i \in \{1,\ldots,k\}} A_i \right| = \sum_{J \subseteq \{1,\ldots,k\}} (-1)^{|J|} \left| \bigcap_{i \in J} \overline{A_i} \right|,$$

where $\overline{A_i} = U \setminus A_i$ and $\bigcap_{i \in \emptyset} = U$.

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Proof sketch.

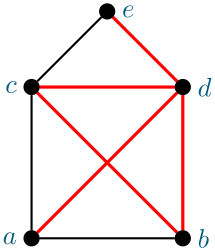
- An element $e \in \bigcap_{i \in \{1,\ldots,k\}} A_i$ is counted on the right only for $J = \emptyset$.
- An element $e \notin \bigcap_{i \in \{1,\ldots,k\}} A_i$ is counted on the right for all $J \subseteq I$, where I is the set of indices i such that $e \notin A_i$.
  - counted negatively for each odd-sized $J \subseteq I$, and positively for each even-sized $J \subseteq I$
  - a non-empty set has as many even-sized subsets as odd-sized subsets

$\square$

Outline

Walks and cycles

- A walk of length k in a graph $G = (V, E)$ (short, a k-walk) is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that $v_i v_{i+1} \in E$ for each $i \in \{0, \ldots, k-1\}$.



(a, d, c, b, d, e)

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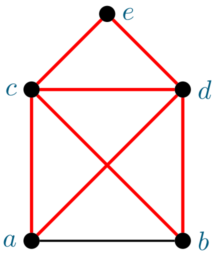
- A walk $(v_0, v_1, \ldots, v_k)$ is closed if $v_0 = v_k$.



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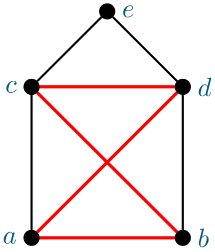
- A walk $(v_0, v_1, \ldots, v_k)$ is closed if $v_0 = v_k$.
- A cycle is a 2-regular subgraph of G.



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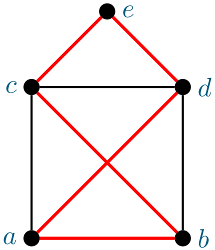
- A walk $(v_0, v_1, \ldots, v_k)$ is closed if $v_0 = v_k$.
- A cycle is a 2-regular subgraph of G.
- A Hamiltonian cycle of G is a cycle of length $n = |V|$.



(a, d, e, c, b, a)

#Hamiltonian-Cycles

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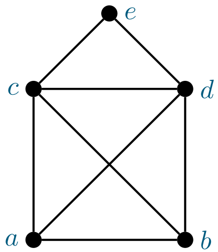
Input: A graph $G = (V, E)$

Output: The number of Hamiltonian cycles of G



This graph has 2 Hamiltonian cycles.

IE for #Hamiltonian-Cycles

- U: the set of closed n-walks starting at vertex 1
- $A_v \subseteq U$: walks in U that visit vertex $v \in V$
- $\Rightarrow$ number of Hamiltonian cycles is $|\bigcap_{v \in V} A_v|$
- To use the IE-theorem, we need to compute $|\bigcap_{v \in S} \overline{A_v}|$, the number of walks from U in the graph $G - S$.

A simpler problem

#Closed n-Walks

Input: An integer n, and a graph $G = (V, E)$ on $\leq n$ vertices

Output: The number of closed n-walks in G starting at vertex 1

A simpler problem

#CLOSED n-WALKS

Input: An integer n, and a graph $G = (V, E)$ on $\leq n$ vertices
Output: The number of closed n-walks in G starting at vertex 1

Dynamic programming

- $T[d, v]$: number of d-walks starting at vertex 1 and ending at vertex v
- Base cases: $T[0, 1] = 1$ and $T[0, v] = 0$ for all $v \in V \setminus \{1\}$
- DP recurrence: $T[d, v] = \sum_{uv \in E} T[d - 1, u]$
- Table T is filled by increasing d
- Return $T[n, 1]$ in $O(n^3)$ time

Wrapping up

- Recall:
  U: set of closed n-walks starting at vertex 1
  A_v: set of closed n-walks that start at vertex 1 and visit vertex v
- By the IE-theorem, the number of Hamiltonian cycles is

$$\left|\bigcap_{v \in V} A_v\right| = \sum_{S \subseteq V} (-1)^{|S|} \left|\bigcap_{v \in S} \overline{A_v}\right|$$

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$$\left|\bigcap_{v\in V} A_v\right| = \sum_{S\subseteq V} (-1)^{|S|} \left|\bigcap_{v\in S} \overline{A_v}\right|$$

- We have seen that $\left|\bigcap_{v\in S} \overline{A_v}\right|$ can be computed in $O(n^3)$ time.
- So, $\sum_{S\subseteq V} (-1)^{|S|} \left|\bigcap_{v\in S} \overline{A_v}\right|$ can be evaluated in $O(2^n n^3)$ time

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Theorem 2

#Hamiltonian-Cycles *can be solved in* $O(2^n n^3)$ *time and polynomial space, where* $n = |V|$.

Outline

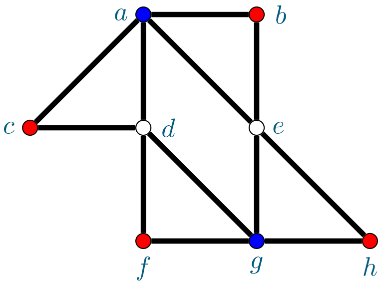
Coloring

A k-coloring of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, ..., k\}$ assigning colors to V such that no two adjacent vertices receive the same color.

Coloring

Input: Graph G, integer k

Question: Does G have a k-coloring?



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Exercise

- Suppose A is an algorithm solving Coloring in $O(f(n))$ time, $n = |V|$, where f is non-decreasing.
- Design a $O^*(f(n))$ time algorithm B, which, for an input graph G, finds a coloring of G with a minimum number of colors.

IE formulation

Observation: partitioning vs. covering

$$G = (V, E) \text{ has a } k\text{-coloring}$$
$$\Leftrightarrow$$
$$G \text{ has independent sets } I_1, \ldots, I_k \text{ such that } \bigcup_{i=1}^{k} I_i = V.$$

IE formulation

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$G = (V, E)$ has a k-coloring

$\Leftrightarrow$

G has independent sets $I_1, \ldots, I_k$ such that $\bigcup_{i=1}^{k} I_i = V$.

- U: set of tuples $(I_1, \ldots, I_k)$, where each I_i, $i \in \{1, \ldots, k\}$, is an independent set

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- Note: $\left|\bigcap_{v \in V} A_v\right| \neq 0 \Leftrightarrow G$ has a k-coloring

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- To use the IE-theorem, we need to compute

$$\left|\bigcap_{v \in S} \overline{A_v}\right| = |\{(I_1, \ldots, I_k) \in U \,:\, I_1, \ldots, I_k \subseteq V \setminus S\}|$$

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$$\left|\bigcap_{v \in S} \overline{A_v}\right| = |\{(I_1, \ldots, I_k) \in U : I_1, \ldots, I_k \subseteq V \setminus S\}|$$

$$= s(V \setminus S)^k,$$

where $s(X)$ is the number of independent sets in $G[X]$

A simpler problem

#IS of Induced Subgraphs

Input: A graph $G = (V, E)$

Output: $s(X)$, the number of independent sets of $G[X]$, for each $X \subseteq V$

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Dynamic Programming

- $s(X)$: the number of independent sets of $G[X]$
- Base case: $s(\emptyset) = 1$
- DP recurrence: $s(X) = s(X \setminus N_G[v]) + s(X \setminus \{v\})$, where $v \in X$
- Table s filled by increasing cardinalities of X
- Output $s(X)$ for each $X \subseteq V$ in time $O^*(2^n)$

Wrapping up

Now, evaluate

$$\left|\bigcap_{v\in V} A_v\right| = \sum_{S\subseteq V} (-1)^{|S|} \left|\bigcap_{v\in S} \overline{A_v}\right| = \sum_{S\subseteq V} (-1)^{|S|} s(V\setminus S)^k,$$

in $O^*(2^n)$ time.
G has a k-coloring iff $\left|\bigcap_{v\in V} A_v\right| > 0$.

Theorem 3 ([Bjørklund & Husfeldt '06], [Koivisto '06])

Coloring can be solved in $O^*(2^n)$ time (and space).

Corollary 4

For a given graph G, a coloring with a minimum number of colors can be found in $O^*(2^n)$ time (and space).

... polynomial space

Using an algorithm by [Gaspers, Lee, 2017], counting all independent sets in a graph on n vertices in $O(1.2355^n)$ time, we obtain a polynomial-space algorithm for $\textsc{Coloring}$ with running time

$$\sum_{S \subseteq V} O(1.2355^{n-|S|}) = \sum_{s=0}^{n} \binom{n}{s} O(1.2377^{n-s}) = O(2.2355^n).$$

Here, we used the Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

Theorem 5

$\textsc{Coloring}$ *can be solved in* $O(2.2355^n)$ *time and polynomial space.*

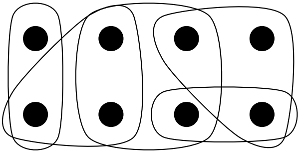
Outline

Counting Set Covers

#Set Covers

Input: A finite ground set V of elements, a collection H of subsets of V, and an integer k

Output: The number of ways to choose a k-tuple of sets $(S_1, \ldots, S_k)$ with $S_i \in H$, $i \in \{1, \ldots, k\}$, such that $\bigcup_{i=1}^{k} S_i = V$.



This instance has $1 \cdot 3! = 6$ covers with 3 sets and $3 \cdot 4! = 72$ covers with 4 sets.

We consider, more generally, that H is given only implicitly, but can be enumerated in $O^*(2^n)$ time and space.

Algorithm for Counting Set Covers

- U: set of k-tuples $(S_1, \dots, S_k)$, where $S_i \in H$, $i \in \{1, \dots, k\}$,
- $A_v = \{(S_1, \dots, S_k) \in U : v \in \bigcup_{i \in \{1,\dots,k\}} S_i\}$,
- the number of covers with k sets is

$$\left| \bigcap_{v \in V} A_v \right| = \sum_{S \subseteq V} (-1)^{|S|} \left| \bigcap_{v \in S} \overline{A_v} \right| = \sum_{S \subseteq V} (-1)^{|S|} s(V \setminus S)^k,$$

where $s(X)$ is the number of sets in H that are subsets of X.

Compute $s(X)$

For each $X \subseteq V$, compute $s(X)$, the number of sets in H that are subsets of X.

Dynamic Programming

- Arbitrarily order $V = \{v_1, v_2, \ldots, v_n\}$
- $g[X,i] = |\{S \in H : (X \cap \{v_i, \ldots, v_n\}) \subseteq S \subseteq X\}|$
- Note: $g[X, n+1] = s(X)$
- Base case: $g[X,1] = \begin{cases} 1 & \text{if } X \in H \\ 0 & \text{otherwise.} \end{cases}$
- DP recurrence: $g[X,i] = \begin{cases} g[X, i-1] & \text{if } v_{i-1} \notin X \\ g[X \setminus \{v_{i-1}\}, i-1] + g[X, i-1] & \text{otherwise.} \end{cases}$
- Table filled by increasing i

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Theorem 6

#SET COVERS *can be solved in* $O^*(2^n)$ *time and space, where* $n = |V|$.

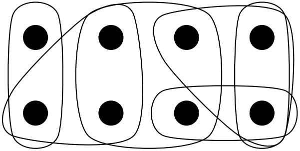
Outline

Counting Set Partitions

#ORDERED SET PARTITIONS

Input: A finite ground set V of elements, a collection H of subsets of V, and an integer k

Output: The number of ways to choose a k-tuple of pairwise disjoint sets $(S_1, \ldots, S_k)$ with $S_i \in H$, $i \in \{1, \ldots, k\}$, such that $\bigcup_{i=1}^{k} S_i = V$. (Now, $S_i \cap S_j = \emptyset$, if $i \neq j$.)



This instance has $1 \cdot 3! = 6$ ordered partitions with 3 sets.

Algorithm

Using a similar approach:

Theorem 7

#Ordered Set Partitions *can be solved in* $O^*(2^n)$ *time and space.*

Algorithm

Using a similar approach:

Theorem 7

#Ordered Set Partitions can be solved in $O^*(2^n)$ time and space.

Corollary 8

There is an algorithm computing the number of k-colorings of an input graph on n vertices in $O^*(2^n)$ time and space.

Covering and partitioning in polynomial space

Theorem 9

The number of covers with k sets and the number of ordered partitions with k sets of a set system (V, H) can be computed in polynomial space and

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Covering and partitioning in polynomial space

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The number of covers with k sets and the number of ordered partitions with k sets of a set system (V, H) can be computed in polynomial space and

1. $O^*(2^n|H|)$ *time, assuming that H can be enumerated in $O^*(|H|)$ time and polynomial space*
2. $O^*(3^n)$ *time, assuming membership in H can be decided in polynomial time, and*

Covering and partitioning in polynomial space

Theorem 9

The number of covers with k sets and the number of ordered partitions with k sets of a set system (V, H) can be computed in polynomial space and

1. $O^*(2^n|H|)$ *time, assuming that H can be enumerated in $O^*(|H|)$ time and polynomial space*
2. $O^*(3^n)$ *time, assuming membership in H can be decided in polynomial time, and*
3. $\sum_{j=0}^{n} \binom{n}{j} T_H(j)$ *time, assuming there is a $T_H(j)$ time and polynomial space algorithm to count for any $W \subseteq V$ with $|W| = j$ the number of sets $S \in H$ satisfying $S \cap W = \emptyset$.*

Outline

Reading

- Chapter 4, *Inclusion-Exclusion* in
  Fedor V. Fomin and Dieter Kratsch. Exact Exponential Algorithms. Springer, 2010.
- Thore Husfeldt. Invitation to Algorithmic Uses of Inclusion-Exclusion. Proceedings of the 38th International Colloquium on Automata, Languages and Programming (ICALP 2011): 42-59, 2011.

Advanced Reading

- Chapter 7, *Subset Convolution* in
  Fedor V. Fomin and Dieter Kratsch. Exact Exponential Algorithms. Springer, 2010.