Social Choice

COMP4418 Knowledge Representation and Reasoning

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Outline

Outline

Social Choice

Social Choice is the theory of aggregating preferences of agents into a socially desirable outcome.

- Mostly studied in Economics and Political Science
- Now also studied within Computer Science (computational social choice)

Applications

- **Search Engines**: to determine the most important sites based on links.
- **Recommender Systems**: to recommend a product to a user based on ratings by users.
- **Multiagent Systems**: to coordinate the actions of groups of autonomous software agents.
- **Fair Division**: allocation of resources.

Outline

Social Choice

- Plurality is the most common voting rule — it selects the alternative that is ranked first by most voters.
- Why do we study different voting rules when we have the *plurality* voting rule?

$$\begin{aligned}
4 \text{ voters } &: a \succ b \succ c \succ d \\
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7 \text{ voters } &: b \succ d \succ c \succ a \\
7 \text{ voters } &: c \succ b \succ d \succ a
\end{aligned}$$

Alternative a is the plurality winner. However plurality may not be the best rule. Why?

Social Choice

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Alternative a is the plurality winner. However plurality may not be the best rule. Why?

In the example

- a majority of voters think a is the worst alternative.
- b, c, and d are better than a in pairwise majority comparisons.

This motivates a study of different voting rules.

Voting Rules

- **Plurality**: alternatives that are ranked first by most voters win.
- **Borda**: Most preferred alternative gets $m - 1$ points, the second most-preferred $m - 2$ points, etc. Alternatives with highest total score win.
- **Plurality with runoff**: Two alternatives that are ranked first by most voters are short-listed. Then among the shortlisted alternatives, the alternative which is preferred by a majority wins.
- **Instant Runoff**: Alternatives that are ranked first by the lowest number of voters are removed from consideration. Repeat until no more alternatives can be deleted.

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Question: which voting rule is used to elect members of the Australian House of Representatives?

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Answer: Instant Runoff

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\end{aligned}
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- Plurality winner: a
- Plurality with runoff: e (after beating a)
- Instant Runoff: d (removal: c, b, e, a)

Borda

Jean-Charles, chevalier de Borda (1733 – 1799): French mathematician, physicist, political scientist, and sailor.

Outline

Formal Framework

Setting:

- Set of agents/voters $N = \{1, \ldots, n\}$
- Set of alternatives $A = \{a_1, \ldots, a_m\}$
- $\succ = (\succ_1, \ldots, \succ_n)$ profile of preferences where each preference $\succ_i$ is a linear order over A.

We denote by $\mathcal{L}(A)$ the set of linear orders over A.

Outcome:

- single selected alternative
- collective preference
- set of collective preference
- a subset of selected alternatives
- probability distribution over alternatives

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Example (Voting Setting)

$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$

$$1 : a \succ_1 b \succ_1 c \succ_1 d$$
$$2 : a \succ_2 c \succ_2 b \succ_2 d$$
$$3 : b \succ_3 d \succ_3 c \succ_3 a$$

Formal Framework

- **Social Welfare Function (SWF):** aggregates individual preferences into a collective preference.

$$F : \mathcal{L}(A)^n \to \mathcal{L}(A)$$

- **Social Preference Function (SPF):** aggregates individual preferences into a set of collective preferences.

$$F : \mathcal{L}(A)^n \to 2^{\mathcal{L}(A)}$$

- **Social Choice Function (SCF):** aggregates individual preferences into a single selected alternative.

$$F : \mathcal{L}(A)^n \to A$$

- **Social Choice Correspondence (SCC):** aggregates individual preferences into a subset of selected alternatives.

$$F : \mathcal{L}(A)^n \to 2^A$$

- **Social Decision Scheme (SDS):** aggregates individual preferences into a probability distribution over alternatives.

$$F : \mathcal{L}(A)^n \to \Delta(A)$$

Formal Framework

Voting rule is an informal term used for social choice functions, social welfare functions etc.

- A social welfare function can be used as social choice function: select the first alternative of the output ranking.
- A social choice correspondence can be used as a social choice function: use some tie-breaking rule over the set of alternatives selected.

Some classes of Voting Rules

- **Based on majority pairwise comparisons**
  - An alternative x wins a pairwise majority comparison against alternative y if $|\{i \in N \mid x \succ_i y\}| > |\{i \in N \mid y \succ_i x\}|$. We can also say that x is preferred over y.
- **Based on weighted majority pairwise comparisons**
  - For any two alternatives $x, y \in A$, the weighted majority pairwise comparison for (x, y) is $|\{i \in N \mid x \succ_i y\}| - |\{i \in N \mid y \succ_i x\}|$.
- **Positional scoring rules** (next slide).

Positional Scoring Rules

A **positional scoring rule (PSR)** is given by a scoring vector $s = (s_1, \ldots, s_m)$ with $s_1 \geq s_2 \cdots \geq s_m$ and $s_1 > s_m$. When a voter puts alternative a in position j, a gets score s_j. Alternatives with the maximum total score win.

- **Borda** rule: PSR with scoring vector $(m-1, m-2, \ldots, 0)$
- **Plurality** rule: PSR with scoring vector $(1, 0, \ldots, 0)$
- **Antiplurality** rule: PSR with scoring vector $(1, 1, \ldots, 1, 0)$
- k **approval** rule: PSR with scoring vector $(\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0)$

Outline

Axioms of Voting Rules

- **Anonymity:** The voting rule treats voters equally: the outcome remains the same as long as the set of votes is the same.
- **Neutrality:** The voting rule treats alternatives equally: F is neutral if $F(\pi(\succ)) = \pi(F(\succ))$ where π is a permutation $\pi : A \longrightarrow A$.
- **Monotonicity:** A chosen alternative will still be chosen when it rises in individual preference rankings (while leaving everything else unchanged).
- **Pareto optimality:** An alternative will not be chosen if there exists another one that all voters prefer the latter to the former.
- **Independence of Irrelevant Alternatives (IIA):** If alternative a is socially preferred to b, then this should not change when a voter changes her ranking of $c \neq a, b$.
- **Non-dictatorial:** there exists no voter such that the outcome is always identical to the preference supplied by the dictator.
- **Condorcet-extension:** if an alternative is pairwise preferred by a majority of voters over every other alternative, then that alternative is selected.
- **Strategyproof:** A voter cannot misreport his/her preference to select a more preferred alternative.

Axioms of Voting Rules

- **Anonymity:** The voting rule treats voters equally: the outcome remains the same as long as the set of votes is the same.

Example

$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$

$$1 : a \succ_1 b \succ_1 c \succ_1 d$$
$$2 : a \succ_2 c \succ_2 b \succ_2 d$$
$$3 : b \succ_3 d \succ_3 c \succ_3 a$$

$$3 : a \succ_1 b \succ_1 c \succ_1 d$$
$$2 : a \succ_2 c \succ_2 b \succ_2 d$$
$$1 : b \succ_3 d \succ_3 c \succ_3 a$$

An anonymous voting rule should have the same output for both preference profiles.

Axioms of Voting Rules

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$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$

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$$3 : b \succ_1 c \succ_1 d \succ_1 a$$
$$2 : b \succ_2 d \succ_2 c \succ_2 a$$
$$1 : c \succ_3 a \succ_3 d \succ_3 b$$

If a neutral voting rule returns a for the first profile, it should return b for the second one.

Axioms of Voting Rules

- **Monotonicity:** A chosen alternative will still be chosen when it rises in individual preference rankings (while leaving everything else unchanged).

Example

$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$

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$$3 : b \succ_3 d \succ_3 a \succ_3 c$$

If a monotonic voting rule returns a for the first profile, it should return a for the second one.

Axioms of Voting Rules

- **Pareto optimality:** An alternative will not be chosen if there exists another one that all voters prefer the latter to the former.

Example

$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$

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A Pareto optimal voting rule will not select b because b is Pareto dominated by a.

Axioms of Voting Rules

- **Independence of Irrelevant Alternatives (IIA):** If alternative a is socially preferred to b, then this should not change when a voter changes her ranking of $c \neq a, b$.

Example

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If a is socially preferred over b, then it should still be socially preferred over b, if ranking of d is changed.

Axioms of Voting Rules

- **Condorcet-extension:** if an alternative is pairwise preferred by a majority of voters over every other alternative, then that alternative is selected.

Example

$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$

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A Condorcet-extension voting rule should select a.

Axioms of Voting Rules

- **Strategyproof:** A voter cannot misreport his/her preference to select a more preferred alternative.

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$N = \{1, 2, 3\}$, $A = \{a, b, c, d\}$

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If the voting rule selects a for the first profile and c for the second profile, it is not strategyproof because 2 can manipulate.

Condorcet's Paradox

Condorcet winner: an alternative that is pairwise preferred by a majority of voters over every other alternative.

There many not exist a Condorcet winner.

Example (Condorcet's Paradox)

$$1 : a \succ_1 b \succ_1 c$$
$$2 : b \succ_2 c \succ_2 a$$
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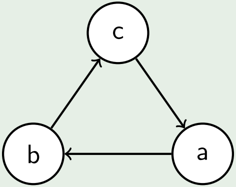

Condorcet

Marquis de Condorcet (1743 – 1794): French philosopher, mathematician, and early political scientist.

Axiomatic method

Formal approach

- **Characterisation Theorems:** show that a particular (class of) rules is the only one satisfying a given set of axioms.
- **Impossibility Theorems:** show which sets of axioms are incompatible.

Outline

Majority Graph

Given $(N, A \succ)$, the **corresponding majority graph** is a directed graph (V, E) in which $(x, y) \in E$ if and only if x is preferred over y by a majority of voters. If $(x, y) \in E$, we say that x *dominates* y. We will denote $D(x) = \{y \mid (x, y) \in E\}$.

Assuming there is an odd number of voters, the pairwise majority graph is a tournament (complete and asymmetric graph).

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Example (Tournament)

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What is the induced majority tournament?

Majority Graph

Given a social choice setting $(N, A \succ)$, the **corresponding majority graph** is a directed graph (V, E) in which $V = A$ and $(x, y) \in E$ if and only if x is preferred over y by a majority of voters. If $(x, y) \in E$, we say that x *dominates* y. We will denote $D(x) = \{y \mid (x, y) \in E\}$.

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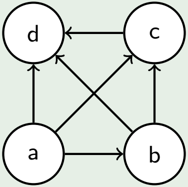

Copeland Rule

The Copeland rule selects alternatives based on the number of other alternatives they dominate. Define the **Copeland score** of an alternative x in tournament $T = (V, E)$ as the outdegree of the alternative.

The set of **Copeland winners** $CO(T)$ then consists of all alternatives that have maximal Copeland score.

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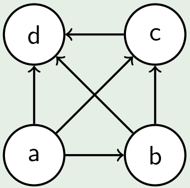
Example (Tournament)

$CO(T) = ?$



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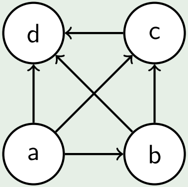
Example (Tournament)

$CO(T) = \{a\}$.



Top Cycle

A non-empty subset $X \subseteq V$ of alternatives in a tournament (V, E) is **dominant** if every alternative in X dominates every alternative outside X.

The **top cycle** of a tournament $T = (V, E)$, denoted by $TC(T)$, is the unique **minimal** dominant subset of V.

Alternative characterization: An alternative is in the top cycle iff it can reach every other alternative by a path in the tournament.

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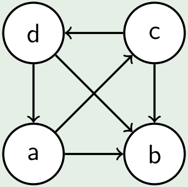
Example (Tournament)

Top cycle=?



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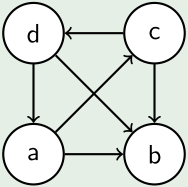
Example (Tournament)

Top cycle: $\{a, c, d\}$



Uncovered Set

The **Uncovered Set** of a tournament $T = (V, E)$, denoted by $UC(T)$, is the set of alternative that can reach every other alternative in at most two steps.

The alternatives in the uncovered set are also referred to as kings.

The uncovered set rule is a Condorcet-extension.

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Example (Tournament)

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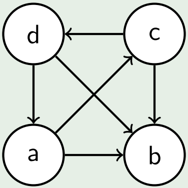

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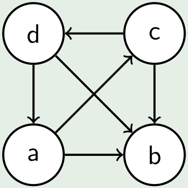
Example (Tournament)

Uncovered Set: $\{a, c, d\}$



Banks

Under the Banks rule, an alternative x is a **Banks winner** if it is a top element in a maximal acyclic subgraph of the tournament. The set of Banks winners of a tournament T is denoted by $BA(T)$.

Computing some Banks winner is easy: grow the set of alternative as long as the graph is acyclic. The top element of the set is a Banks winner.

Theorem (Woeginger, 2003)

Checking whether a certain alternative is a Banks winner is NP-complete.

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Example (Tournament)

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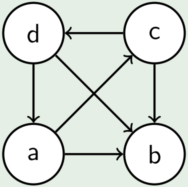


Relations between Tournament Solutions

Theorem

Any Copeland winner is a member of the uncovered set.

$$CO(T) \subseteq UC(T)$$

Proof.

- Supposed that an alternative x is a Copeland winner but not a member of the uncovered set.
- This means that $x \cup D(x) \cup D(D(x)) \neq A$.
- Hence there exists some $y \in A \setminus (\{x\} \cup D(x) \cup D(D(x)))$.
- Thus, $D(y) = \{x\} \cup D(x)$ so that Copeland score of y is more than that of x.

$\square$

Relations between Tournament Solutions

Theorem

Any member of the uncovered set is a member of the top cycle.

$$UC(T) \subseteq TC(T).$$

Proof.

If an alternative is a member of the uncovered set, then it can reach each other alternative in at most two steps so it reaches all other alternatives. □

Relations between Tournament Solutions

Theorem

Any member of the Banks set is a member of the uncovered set.

$$BA(T) \subseteq UC(T)$$

Proof.

- Consider a Banks winner b which is the the top element of a maximal acyclic subgraph of the tournament with vertex set V'. Note that b dominates each vertex in $V' \setminus \{b\}$.
- Consider any vertex $v \in V \setminus V'$.
- Then the graph induced by $V' \cup \{v\}$ is not acyclic which means that there exist $x, y \in V'$ such that and $(x, y) \in E$, $(y, v) \in E$ and $(v, x) \in E$.
  - If $b = y$, then b dominates y.
  - If $b \neq y$, b dominates y which dominates v.

In either case, b reaches v in at most two steps.

□

Outline

May's Theorem

- **Positive reinforcement:** A voting rule satisfies positive reinforcement if, whenever some voter reinforces a (possibly tied) winner x in his list, then x will become the unique winner.

Theorem (May, 1952)

A voting rule for two alternatives satisfies anonymity, neutrality, and positive reinforcement if and only if it is the plurality rule.

Positional Scoring Rules are not Condorcet-extensions

Theorem (Condorcet, 1785)

Borda's rule is not a Condorcet-extension when there are 3 or more alternatives.

Theorem (Fishburn, 1973)

No positional scoring rule is a Condorcet-extension when there are 3 or more alternatives.

Consider the following profile:

$$\begin{aligned} 6 \text{ voters } &: a \succ b \succ c \\ 3 \text{ voters } &: c \succ a \succ b \\ 4 \text{ voters } &: b \succ a \succ c \\ 4 \text{ voters } &: b \succ c \succ a \end{aligned}$$

- Score of a: $6s_1 + 7s_2 + 4s_3$
- Score of b: $8s_1 + 6s_2 + 3s_3$
- Score of c: $3s_1 + 4s_2 + 10s_3$

Alternative b is the winner under every PSR. However a is the Condorcet winner.

Arrow's Theorem

Theorem (Arrow's Theorem)

Any SWF for three or more alternatives cannot satisfy all the three axioms:

1. *Pareto optimality*
2. *Independence of Irrelevant Alternatives (IIA)*
3. *Non-dictatorship*

Muller-Satterthwaite Theorem

- **Strong Monotonicity:** It states that if an alternative x is a winner in a preference profile P and if a preference profile Q is constructed so that x's position remains the same or improves with respect to all other alternatives, then x is the winner in Q as well.

Theorem (Muller and Satterthwaite, 1977)

Any social choice function for 3 alternatives that is onto and strongly monotonic must be a dictatorship.

Gibbard–Satterthwaite Theorem

Theorem (Gibbard–Satterthwaite Theorem)

Any social choice function for three or more alternatives cannot satisfy all the three axioms:

1. *Onto*
2. *Strategyproofness*
3. *Non-dictatorship*

Young's characterization of positional scoring rules

- **Reinforcement:** A voting rule satisfies reinforcement if, whenever we split the electorate into two groups and some alternative would win in both groups, then it will also win for the full electorate.
- **Continuity:** A voting rule is continuous if, whenever electorate N elects a unique winner x, then for any other electorate N' there exists a number k s.t. N' together with k copies of N will also elect only x.

Theorem (Young, 1975)

A voting rule satisfies anonymity, neutrality, reinforcement, and continuity iff it is a positional scoring rule.

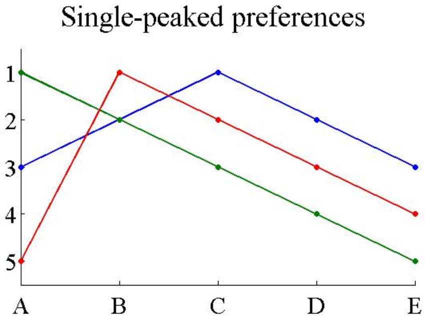
Outline

Domain Restriction

A preference profile has **single-peaked preferences** if there exists a left to right ordering $>$ on the alternatives such that any voter prefers a to b if a is between b and her top alternative.

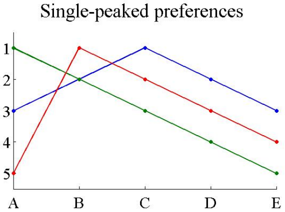
Examples

- Airconditioner temperature.
- Political spectrum.



Domain Restriction

Given a left-to-right ordering $>$, the **median voter rule** asks each voter for their top alternative and elects the alternative proposed by the voter corresponding to the median with respect to $>$.



Domain Restriction

Given a left-to-right ordering $>$, the **median voter rule** asks each voter for their top alternative and elects the alternative proposed by the voter corresponding to the median with respect to $>$.

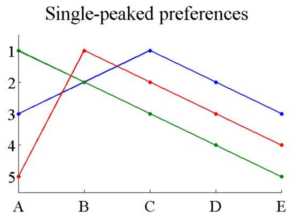
Theorem (Black's Theorem, 1948)

If an odd number of voters submit single-peaked preferences, then there exists a Condorcet winner and it will get elected by the median voter rule.

$$1 : A \succ B \succ C \succ D \succ E$$

$$2 : B \succ C \succ D \succ E \succ A$$

$$3 : C \succ B \succ D \succ A \succ E$$



Outline

Social Decision Schemes

Social Decision Scheme: aggregates individual preferences into a probability distribution over alternatives.

$$F : \mathcal{L}(A)^n \to \Delta(A)$$

Social Decision Schemes

Random Dictatorship: probability of an alternative is proportional to its plurality score.

Example (Random Dictatorship)

$$
\begin{aligned}
1: &\quad a \succ b \succ c \\
2: &\quad b \succ a \succ c \\
3: &\quad b \succ c \succ a
\end{aligned}
$$

Social Decision Schemes

Random Dictatorship: probability of an alternative is proportional to its plurality score.

Example (Random Dictatorship)

$$
\begin{aligned}
1: &\quad a \succ b \succ c \\
2: &\quad b \succ a \succ c \\
3: &\quad b \succ c \succ a
\end{aligned}
$$

$$p(a) = 1/3, p(b) = 2/3, p(c) = 0.$$

Social Decision Schemes

Random Dictatorship: probability of an alternative is proportional to its plurality score.

Theorem

Random dictatorship is the only social decision scheme that is anonymous, strategyproof, and has Pareto optimal alternatives in the support.

Social Decision Schemes

Borda Proportional: probability of an alternative is proportional to its Borda score.

Example (Borda Proportional)

$$
\begin{aligned}
1 : &\quad a \succ b \succ c \\
2 : &\quad b \succ a \succ c \\
3 : &\quad b \succ c \succ a
\end{aligned}
$$

Social Decision Schemes

Borda Proportional: probability of an alternative is proportional to its Borda score.

Example (Borda Proportional)

$$\begin{aligned} 1: &\quad a \succ b \succ c \\ 2: &\quad b \succ a \succ c \\ 3: &\quad b \succ c \succ a \end{aligned}$$

Borda score of a is 3; Borda score of b is 5; Borda score of c is 1;

$$p(a) = 3/9, p(b) = 5/9, p(c) = 1/9.$$

Social Decision Schemes

Borda Proportional: probability of an alternative is proportional to its Borda score.

Theorem

Borda Proportional is strategyproof but may give Pareto dominated alternatives non-zero probability.

Outline

Kemeny

- A Kemeny ranking is

$$\arg\min_{L \in \mathcal{L}} \sum_{x,y \in A, x \succ y} |\{i \in N \mid y \succ_i x\}|$$

- A Kemeny winner is the maximal alternative of a Kemeny ranking.

In order to order to find a Kemeny winner, it is enough to examine the **weighted majority graph** which is a weighted graph on A where for each $a, b \in A$, the weight $w(x, y)$ is $|\{i \in N \mid x \succ_i y\}| - |\{i \in | \ y \succ_i x\}|$.

An optimal ranking one which can be obtained from the weighted majority graph with the minimum total weight of arcs that is flipped.

The Kemeny rule is a Condorcet-extension because we do not need to flip arcs for the Condorcet winner.

Theorem (Bartholdi et al., 1989)

Finding a Kemeny ranking and a Kemeny winner is NP-hard.

Other Voting Rules

- **Approval Voting**: Each voter approves a subset of alternatives. Alternatives that are approved by most voters win.
- **Range Voting**: Voters assign up to 100 points to each alternative. Alternatives with the highest total scores win.
- **Sequential majority comparisons**: Alternatives that win a sequence of pairwise comparisons win.

Borda versus Condorcet

- Jean-Charles, chevalier de Borda (1733 – February 1799): French mathematician, physicist, political scientist, and sailor.
- Marquis de Condorcet (1743 – 1794): French philosopher, mathematician, and early political scientist

Combining Borda and Condorcet:

- **Black's rule:** Return the Condorcet winner if one exists and the Borda winner otherwise.
- **Nanson's rule:** Runoff rule in which alternatives with the lowest Borda scores are deleted until no more deletions are possible.

Summary

- There are many interesting voting rules with relative merits.
- An axiomatic study of voting rules helps understand the relative merits.
- Voting rules have different computational complexity as well.
- Social choice has many famous impossibility results.
- Considering restricted domains and randomisation are two possible ways to avoid impossibility results.

Outline

Reading

Social choice chapters of the following books:

- Y. Shoham and K. Leyton-Brown. Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations. 2009. `http://www.masfoundations.org`
- N. Nisan, T. Roughgarden, E. Tardos, and V.V. Vazirani. Algorithmic Game Theory. Cambridge University Press, 2007. `www.cambridge.org/journals/nisan/downloads/Nisan_Non-printable.pdf`

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