5b. Measure & Conquer

COMP6741: Parameterized and Exact Computation

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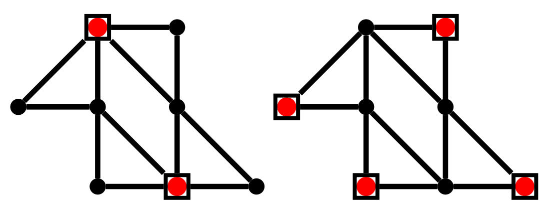
19T3

Contents

1 Introduction

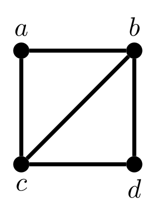
Recall: Maximal Independent Sets

- A vertex set $S \subseteq V$ of a graph $G = (V, E)$ is an *independent set* in G if there is no edge $uv \in E$ with $u, v \in S$.
- An independent set is *maximal* if it is not a subset of any other independent set.
- Examples:



Enumeration problem: Enumerate all maximal independent sets

Enum-MIS	
Input:	graph G
Output:	all maximal independent sets of G



Maximal independent sets: $\{a, d\}, \{b\}, \{c\}$

Note: Let v be a vertex of a graph G. Every maximal independent set contains a vertex from $N_G[v]$.

Branching Algorithm for Enum-MIS

Algorithm ***enum-mis***(G, I)
Input : A graph $G = (V, E)$, an independent set I of G.
Output: All maximal independent sets of G that are supersets of I.

$G' \leftarrow G - N_G[I]$
if $V(G') = \emptyset$ **then** // G' has no vertex
  Output I
else
  Select $v \in V(G')$ such that $d_{G'}(v) = \delta(G')$ // v has min degree in G'
  Run enum-mis$(G, I \cup \{u\})$ for each $u \in N_{G'}[v]$

Running Time Analysis

Let us upper bound by $L(n) = 2^{\alpha n}$ the number of leaves in any search tree of **enum-mis** for an instance with $|V(G')| \le n$.

We minimize α subject to constraints obtained from the branching:

$$
\begin{aligned}
& L(n) \ge (d+1) \cdot L(n-(d+1)) && \text{for each integer } d \ge 0.\\
\Leftrightarrow\quad & 2^{\alpha n} \ge d' \cdot 2^{\alpha \cdot (n-d')} && \text{for each integer } d' \ge 1.\\
\Leftrightarrow\quad & 1 \ge d' \cdot 2^{\alpha \cdot (-d')} && \text{for each integer } d' \ge 1.
\end{aligned}
$$

For fixed d', the smallest value for 2^α satisfying the constraint is $d'^{1/d'}$. The function $f(x) = x^{1/x}$ has its maximum value for $x = e$ and for integer x the maximum value of $f(x)$ is when $x = 3$.

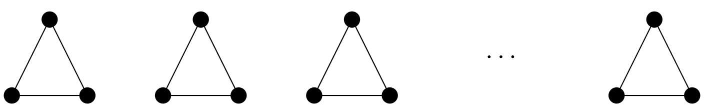
Therefore, the minimum value for 2^α for which all constraints hold is $3^{1/3}$. We can thus set $L(n) = 3^{n/3}$.

Since the height of the search trees is $\le |V(G')|$, we obtain:

Theorem 1. *Algorithm **enum-mis** has running time $O^*(3^{n/3}) \subseteq O(1.4423^n)$, where $n = |V|$.*

Corollary 2. *A graph on n vertices has $O(3^{n/3})$ maximal independent sets.*

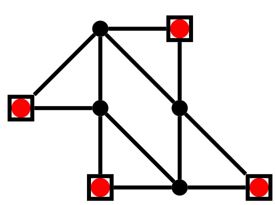
Running Time Lower Bound



Theorem 3. *There is an infinite family of graphs with $\Omega(3^{n/3})$ maximal independent sets.*

2 Maximum Independent Set

MAXIMUM INDEPENDENT SET	
Input:	graph G
Output:	A largest independent set of G.



Branching Algorithm for Maximum Independent Set

Algorithm ***mis***(G)
Input : A graph $G = (V, E)$.
Output: The size of a maximum i.s. of G.

1 **if** $\Delta(G) \leq 2$ **then** // G has max degree ≤ 2
2   **return** the size of a maximum i.s. of G in polynomial time
3 **else if** $\exists v \in V : d(v) = 1$ **then** // v has degree 1
4   **return** $1 + \textbf{mis}(G - N[v])$
5 **else if** G is not connected **then**
6   Let G_1 be a connected component of G
7   **return** $\textbf{mis}(G_1) + \textbf{mis}(G - V(G_1))$
8 **else**
9   Select $v \in V$ s.t. $d(v) = \Delta(G)$ // v has max degree
10   **return** $\max(1 + \textbf{mis}(G - N[v]), \textbf{mis}(G - v))$

Correctness

Line 4:

Lemma 4. *If $v \in V$ has degree 1, then G has a maximum independent set I with $v \in I$.*

Proof. Let J be a maximum independent set of G. If $v \in J$ we are done because we can take $I = J$. If $v \notin J$, then $u \in J$, where u is the neighbor of v, otherwise J would not be maximum. Set $I = (J \setminus \{u\}) \cup \{v\}$. We have that I is an independent set, and, since $|I| = |J|$, I is a maximum independent set containing v. $\square$

2.1 Simple Analysis

Lemma 5 (Simple Analysis Lemma)**.** *Let*

- *A be a branching algorithm*
- *$\alpha > 0$, $c \geq 0$ be constants*

such that on input I, A calls itself recursively on instances $I_1, \ldots, I_k$, but, besides the recursive calls, uses time $O(|I|^c)$, such that

$$(\forall i : 1 \leq i \leq k) \quad |I_i| \leq |I| - 1, \text{ and} \tag{1}$$
$$2^{\alpha \cdot |I_1|} + \cdots + 2^{\alpha \cdot |I_k|} \leq 2^{\alpha \cdot |I|}. \tag{2}$$

Then A solves any instance I in time $O(|I|^{c+1}) \cdot 2^{\alpha \cdot |I|}$.

Proof. By induction on $|I|$. W.l.o.g., suppose the hypotheses' O statements hide a constant factor $d \geq 0$, and for the base case assume that the algorithm returns the solution to an empty instance in time $d \leq d \cdot |I|^{c+1} 2^{\alpha \cdot |I|}$.

Suppose the lemma holds for all instances of size at most $|I| - 1 \geq 0$, then the running time of algorithm A on instance I is

$$\begin{aligned}
T_A(I) &\leq d \cdot |I|^c + \sum_{i=1}^{k} T_A(I_i) && \text{(by definition)} \\
&\leq d \cdot |I|^c + \sum d \cdot |I_i|^{c+1} 2^{\alpha \cdot |I_i|} && \text{(by the inductive hypothesis)} \\
&\leq d \cdot |I|^c + d \cdot (|I| - 1)^{c+1} \sum 2^{\alpha \cdot |I_i|} && \text{(by (1))} \\
&\leq d \cdot |I|^c + d \cdot (|I| - 1)^{c+1} 2^{\alpha \cdot |I|} && \text{(by (2))} \\
&\leq d \cdot |I|^{c+1} 2^{\alpha \cdot |I|}.
\end{aligned}$$

The final inequality uses that $\alpha \cdot |I| > 0$ and holds for any $c \geq 0$. $\square$

Simple Analysis for mis

- At each node of the search tree: $O(n^2)$ time
- G disconnected: (1) If $\alpha \cdot s < 1$, then $s < 1/\alpha$, and the algorithm solves G_1 in constant time (provided that $\alpha > 0$). We can view this rule as a simplification rule, removing G_1 and making one recursive call on $G - V(G_1)$. (2) If $\alpha \cdot (n-s) < 1$: similar as (1). (3) Otherwise,

$$(\forall s : 1/\alpha \le s \le n - 1/\alpha) \quad 2^{\alpha \cdot s} + 2^{\alpha \cdot (n-s)} \le 2^{\alpha \cdot n}. \tag{3}$$

  always satisfied since $2^x + 2^y \le 2^{x+y}$ if $x, y \ge 1$.

- Branch on vertex of degree $d \ge 3$

$$(\forall d : 3 \le d \le n-1) \quad 2^{\alpha \cdot (n-1)} + 2^{\alpha \cdot (n-1-d)} \le 2^{\alpha n}. \tag{4}$$

  Dividing all these terms by $2^{\alpha n}$, the constraints become

$$2^{-\alpha} + 2^{\alpha \cdot (-1-d)} \le 1. \tag{5}$$

Compute optimum α
The minimum α satisfying the constraints is obtained by solving a convex mathematical program minimizing α subject to the constraints (the constraint for $d = 3$ is sufficient as all other constraints are weaker).

Alternatively, set $x := 2^\alpha$, compute the unique positive real root of each of the *characteristic polynomials*

$$c_d(x) := x^{-1} + x^{-1-d} - 1,$$

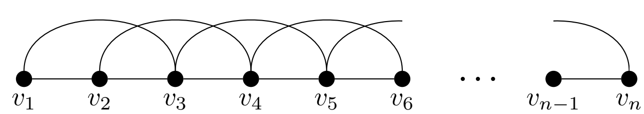
and take the maximum of these roots [Kul99].

d	x	α
3	1.3803	0.4650
4	1.3248	0.4057
5	1.2852	0.3620
6	1.2555	0.3282
7	1.2321	0.3011

Simple Analysis: Result

- use the Simple Analysis Lemma with $c = 2$ and $\alpha = 0.464959$
- running time of Algorithm **mis** upper bounded by $O(n^3) \cdot 2^{0.464959 \cdot n} = O(2^{0.4650 \cdot n})$ or $O(1.3803^n)$

Lower bound



$$T(n) = T(n-5) + T(n-3)$$

- for this graph, P_n^2, the worst case running time is $1.1938\ldots^n \cdot \mathsf{poly}(n)$
- Run time of algo **mis** is $\Omega(1.1938^n)$

Worst-case running time — a mystery

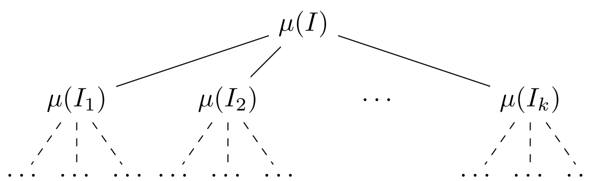
What is the worst-case running time of Algorithm **mis**?

- lower bound $\Omega(1.1938^n)$
- upper bound $O(1.3803^n)$

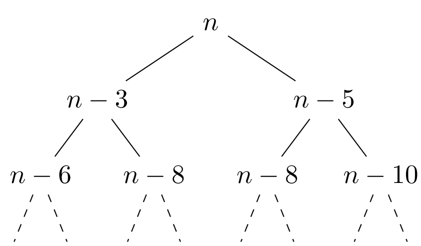
2.2 Search Trees and Branching Numbers

Search Trees

Denote $\mu(I) := \alpha \cdot |I|$.



Example: execution of **mis** on a P_n^2



Branching number: Definition

Consider a constraint

$$2^{\mu(I)-a_1} + \cdots + 2^{\mu(I)-a_k} \leq 2^{\mu(I)}.$$

Its *branching number* is

$$2^{-a_1} + \cdots + 2^{-a_k},$$

and is denoted by

$$(a_1, \ldots, a_k)\,.$$

Clearly, any constraint with branching number at most 1 is satisfied.

Branching numbers: Properties

Dominance For any a_i, b_i such that $a_i \geq b_i$ for all i, $1 \leq i \leq k$,

$$(a_1, \ldots, a_k) \leq (b_1, \ldots, b_k)\,,$$

as $2^{-a_1} + \cdots + 2^{-a_k} \leq 2^{-b_1} + \cdots + 2^{-b_k}$.
In particular, for any $a, b > 0$,

$$\text{either} \quad (a,a) \leq (a,b) \quad \text{or} \quad (b,b) \leq (a,b)\,.$$

Balance If $0 < a \leq b$, then for any ε such that $0 \leq \varepsilon \leq a$,

$$(a,b) \leq (a-\varepsilon, b+\varepsilon)$$

by convexity of 2^x.

2.3 Measure & Conquer Analysis

- Goal
  - capture more structural changes when branching into subinstances
- How?
  - potential-function method, a.k.a., *Measure & Conquer* [FGK09]
- Example: Algorithm **mis**
  - advantage when degrees of vertices decrease

Measure

Instead of using the number of vertices, n, to track the progress of **mis**, let us use a measure μ of G.

Definition 6. A *measure* μ for a problem P is a function from the set of all instances for P to the set of non negative reals.

Let us use the following measure for the analysis of **mis** on graphs of maximum degree at most 5:

$$\mu(G) = \sum_{i=0}^{5} \omega_i n_i,$$

where $n_i := |\{v \in V : d(v) = i\}|$.

Measure & Conquer Analysis

Lemma 7 (Measure & Conquer Lemma)**.** *Let*

- *A be a branching algorithm*
- *$c \geq 0$ be a constant, and*
- *$\mu(\cdot), \eta(\cdot)$ be two measures for the instances of A,*

such that on input I, A calls itself recursively on instances $I_1, \dots, I_k$, but, besides the recursive calls, uses time $O(\eta(I)^c)$, such that

$$(\forall i) \quad \eta(I_i) \leq \eta(I) - 1, \text{ and} \tag{6}$$

$$2^{\mu(I_1)} + \ldots + 2^{\mu(I_k)} \leq 2^{\mu(I)}. \tag{7}$$

Then A solves any instance I in time $O(\eta(I)^{c+1}) \cdot 2^{\mu(I)}$.

Analysis of mis for degree at most 5

For $\mu(G) = \sum_{i=0}^{5} \omega_i n_i$ to be a valid measure, we constrain that

$$w_d \geq 0 \qquad \text{for each } d \in \{0, \dots, 5\}$$

We also constrain that reducing the degree of a vertex does not increase the measure (useful for analysis of the degree-1 simplification rule and the branching rule):

$$-\omega_d + \omega_{d-1} \leq 0 \qquad \text{for each } d \in \{1, \dots, 5\}$$

Lines 1–2 is a halting rule and we merely need that it takes polynomial time so that we can apply Lemma 7. Lines 3–4 of **mis** need to satisfy (7).

The simplification rule removes v and its neighbor u. We get a constraint for each possible degree of u:

$$
\begin{aligned}
& 2^{\mu(G)-\omega_1-\omega_d} \leq 2^{\mu(G)} && \text{for each } d \in \{1,\ldots,5\} \\
\Leftrightarrow \quad & 2^{-\omega_1-\omega_d} \leq 2^0 && \text{for each } d \in \{1,\ldots,5\} \\
\Leftrightarrow \quad & -\omega_1 - \omega_d \leq 0 && \text{for each } d \in \{1,\ldots,5\}
\end{aligned}
$$

These constraints are always satisfied since $\omega_d \geq 0$ for each $d \in \{0,\ldots,5\}$. **Note:** the degrees of u's other neighbors (if any) decrease, but this degree change does not increase the measure.

For lines 5–7 of **mis** we consider two cases.

If $\mu(G_1) < 1$ (or $\mu(G - V(G_1)) < 1$, which is handled similarly), then we view this rule as a simplification rule, which takes polynomial time to compute **mis**(G_1), and then makes a recursive call **mis**$(G - V(G_1))$. To ensure that instances with measure < 1 can be solved in polynomial time, we constrain that

$$
w_d > 0 \qquad \text{for each } d \in \{3,4,5\}
$$

and this will be implied by other constraints.

Otherwise, $\mu(G_1) \geq 1$ and $\mu(G - V(G_1)) \geq 1$, and we need to satisfy (7). Since $\mu(G) = \mu(G_1) + \mu(G - V(G_1))$, the constraints

$$
2^{\mu(G_1)} + 2^{\mu(G-V(G_1))} \leq 2^{\mu(G)}
$$

are always satisfied since the slope of the function 2^x is at least 1 when $x \geq 1$. (I.e., we get no new constraints on $\omega_1,\ldots,\omega_5$.)

Lines 8–10 of **mis** need to satisfy (7). We know that in $G - N[v]$, some vertex of $N^2[v]$ has its degree decreased (unless G has at most 6 vertices, which can be solved in constant time). Define

$$
(\forall d : 2 \leq d \leq 5) \quad h_d := \min_{2 \leq i \leq d} \{w_i - w_{i-1}\}
$$

We obtain the following constraints:

$$
\begin{aligned}
& 2^{\mu(G)-w_d-\sum_{i=2}^d p_i \cdot (w_i - w_{i-1})} + 2^{\mu(G)-w_d-\sum_{i=2}^d p_i \cdot w_i - h_d} \leq 2^{\mu(G)} \\
\Leftrightarrow \quad & 2^{-w_d-\sum_{i=2}^d p_i \cdot (w_i - w_{i-1})} + 2^{-w_d-\sum_{i=2}^d p_i \cdot w_i - h_d} \leq 1
\end{aligned}
$$

for all $d, 3 \leq d \leq 5$ (degree of v), and all $p_i, 2 \leq i \leq d$, such that $\sum_{i=2}^d p_i = d$ (number of neighbors of degree i).

Applying the lemma

Our constraints

$$
\begin{aligned}
w_d &\geq 0 \\
-\omega_d + \omega_{d-1} &\leq 0 \\
2^{-w_d-\sum_{i=2}^d p_i \cdot (w_i - w_{i-1})} + 2^{-w_d-\sum_{i=2}^d p_i \cdot w_i - h_d} &\leq 1
\end{aligned}
$$

are satisfied by the following values:

i	w_i	h_i
1	0	0
2	0.25	0.25
3	0.35	0.10
4	0.38	0.03
5	0.40	0.02

These values for w_i satisfy all the constraints and $\mu(G) \leq 2n/5$ for any graph of max degree ≤ 5. Taking $c = 2$ and $\eta(G) = n$, the Measure & Conquer Lemma shows that **mis** has run time $O(n^3)2^{2n/5} = O(1.3196^n)$ on graphs of max degree ≤ 5.

2.4 Optimizing the measure

Compute optimal weights

- By convex programming [**GaspersS09**]

All constraints are already convex, except conditions for h_d

$$(\forall d : 2 \leq d \leq 5) \quad h_d := \min_{2 \leq i \leq d} \{w_i - w_{i-1}\}$$
$$\Downarrow$$
$$(\forall i, d : 2 \leq i \leq d \leq 5) \quad h_d \leq w_i - w_{i-1}.$$

Use existing convex programming solvers to find optimum weights.

Convex program in AMPL

```
param maxd integer = 5;
set DEGREES := 0..maxd;
var W {DEGREES} >= 0;  # weight for vertices according to their degrees
var g {DEGREES} >= 0;  # weight for degree reductions from deg i
var h {DEGREES} >= 0;  # weight for degree reductions from deg <= i
var Wmax;              # maximum weight of W[d]

minimize Obj: Wmax;    # minimize the maximum weight

subject to MaxWeight {d in DEGREES}:
  Wmax >= W[d];
subject to gNotation {d in DEGREES : 2 <= d}:
  g[d] <= W[d]-W[d-1];
subject to hNotation {d in DEGREES, i in DEGREES : 2 <= i <= d}:
  h[d] <= W[i]-W[i-1];
subject to Deg3 {p2 in 0..3, p3 in 0..3 : p2+p3=3}:
  2^(-W[3] -p2*g[2] -p3*g[3]) + 2^(-W[3] -p2*W[2] -p3*W[3] -h[3]) <=1;
subject to Deg4 {p2 in 0..4, p3 in 0..4, p4 in 0..4 : p2+p3+p4=4}:
  2^(-W[4] - p2*g[2] - p3*g[3] - p4*g[4])
+ 2^(-W[4] - p2*W[2] - p3*W[3] - p4*W[4] - h[4]) <=1;
subject to Deg5 {p2 in 0..5, p3 in 0..5, p4 in 0..5, p5 in 0..5 :
                 p2+p3+p4+p5=5}:
  2^(-W[5] - p2*g[2] - p3*g[3] - p4*g[4] - p5*g[5])
+ 2^(-W[5] - p2*W[2] - p3*W[3] - p4*W[4] - p5*W[5] - h[5]) <=1;
```

Optimal weights

i	w_i	h_i
1	0	0
2	0.206018	0.206018
3	0.324109	0.118091
4	0.356007	0.031898
5	0.358044	0.002037

- use the Measure & Conquer Lemma with $\mu(G) = \sum_{i=1}^{5} w_i n_i \leq 0.358044 \cdot n$, $c = 2$, and $\eta(G) = n$
- **mis** has running time $O(n^3)2^{0.358044 \cdot n} = O(1.2817^n)$

2.5 Exponential Time Subroutines

Lemma 8 (Combine Analysis Lemma)**.** *Let*

- *A be a branching algorithm and B be an algorithm,*
- *$c \geq 0$ be a constant, and*
- *$\mu(\cdot), \mu'(\cdot), \eta(\cdot)$ be three measures for the instances of A and B,*

such that $\mu'(I) \leq \mu(I)$ for all instances I, and on input I, A either solves I by invoking B with running time $O(\eta(I)^{c+1}) \cdot 2^{\mu'(I)}$, or calls itself recursively on instances $I_1, \ldots, I_k$, but, besides the recursive calls, uses time $O(\eta(I)^c)$, such that

$$(\forall i) \quad \eta(I_i) \leq \eta(I) - 1, \text{ and} \tag{8}$$

$$2^{\mu(I_1)} + \ldots + 2^{\mu(I_k)} \leq 2^{\mu(I)}. \tag{9}$$

Then A solves any instance I in time $O(\eta(I)^{c+1}) \cdot 2^{\mu(I)}$.

Algorithm mis on general graphs

- use the Combine Analysis Lemma with $A = B = \textbf{mis}$, $c = 2$, $\mu(G) = 0.35805n$, $\mu'(G) = \sum_{i=1}^{5} w_i n_i$, and $\eta(G) = n$
- for every instance G, $\mu'(G) \leq \mu(G)$ because $\forall i, w_i \leq 0.35805$
- for each $d \geq 6$,

$$(0.35805, (d+1) \cdot 0.35805) \leq 1$$

- Thus, Algorithm **mis** has running time $O(1.2817^n)$ for graphs of arbitrary degrees

2.6 Structures that arise rarely

Rare Configurations

- Branching on a local configuration C does not influence overall running time if C is selected only a constant number of times on the path from the root to a leaf of any search tree corresponding to the execution of the algorithm
- Can be proved formally by using measure

$$\mu'(I) := \begin{cases} \mu(I) + c & \text{if } C \text{ may be selected in the current subtree} \\ \mu(I) & \text{otherwise.} \end{cases}$$

Avoid branching on regular instances in mis

else
  Select $v \in V$ such that
    (1) v has maximum degree, and
    (2) among all vertices satisfying (1), v has a neighbor of
      minimum degree
  return $\max(1 + \textbf{mis}(G - N[v]), \textbf{mis}(G - v))$

New measure:

$$\mu'(G) = \mu(G) + \sum_{d=3}^{5} [G \text{ has a } d\text{-regular subgraph}] \cdot C_d$$

where $C_d, 3 \leq d \leq 5$, are constants. The Iverson bracket $[F] = \begin{cases} 1 \text{ if } F \text{ true} \\ 0 \text{ otherwise} \end{cases}$

Resulting Branching numbers

For each $d, 3 \leq d \leq 5$ and all $p_i, 2 \leq i \leq d$ such that $\sum_{i=2}^{d} p_i = d$ *and* $p_d \neq d$,

$$\Big(w_d + \sum_{i=2}^{d} p_i \cdot (w_i - w_{i-1}), w_d + \sum_{i=2}^{d} p_i \cdot w_i + h_d\Big).$$

All these branching numbers are at most 1 with the optimal set of weights

Result

i	w_i	h_i
1	0	0
2	0.207137	0.207137
3	0.322203	0.115066
4	0.343587	0.021384
5	0.347974	0.004387

Thus, the modified Algorithm **mis** has running time $O(2^{0.3480 \cdot n}) = O(1.2728^n)$.

Current fastest algorithm for MIS: $O(1.1996^n)$ [XN17]

3 Further Reading

- Chapter 2, *Branching* in [FK10]
- Chapter 6, *Measure & Conquer* in [FK10]
- Chapter 2, *Branching Algorithms* in [Gas10]

References

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