6. Kernelization

COMP6741: Parameterized and Exact Computation

Serge Gaspers[1,2]

[1] School of Computer Science and Engineering, UNSW Australia
[2] Data61, Decision Sciences Group, CSIRO

Semester 2, 2016

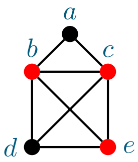
Outline

Outline

Vertex cover

A vertex cover of a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that for each edge $\{u, v\} \in E$, we have $u \in S$ or $v \in S$.

VERTEX COVER	
Input:	A graph $G = (V, E)$ and an integer k
Parameter:	k
Question:	Does G have a vertex cover of size at most k?



Outline

Simplification rules for Vertex Cover

(Degree-0)

If $\exists v \in V$ such that $d_G(v) = 0$, then set $G \leftarrow G - v$.

Simplification rules for VERTEX COVER

(Degree-0)

If $\exists v \in V$ such that $d_G(v) = 0$, then set $G \leftarrow G - v$.

Proving correctness. A simplification rule is sound if for any instance, it produces an equivalent instance. Two instances I, I' are equivalent if they are both YES-instances or they are both NO-instances.

Lemma 1

(Degree-0) is sound.

Simplification rules for Vertex Cover

(Degree-0)

If $\exists v \in V$ such that $d_G(v) = 0$, then set $G \leftarrow G - v$.

Proving correctness. A simplification rule is sound if for any instance, it produces an equivalent instance. Two instances I, I' are equivalent if they are both Yes-instances or they are both No-instances.

Lemma 1

(Degree-0) is sound.

Proof.

First, suppose $(G - v, k)$ is a Yes-instance. Let S be a vertex cover for $G - v$ of size at most k. Then, S is also a vertex cover for G since no edge of G is incident to v. Thus, (G, k) is a Yes-instance.

Now, suppose (G, k) is a Yes-instance. For the sake of contradiction, assume $(G - v, k)$ is a No-instance. Let S be a vertex cover for G of size at most k. But then, $S \setminus \{v\}$ is a vertex cover of size at most k for $G - v$; a contradiction. □

Simplification rules for Vertex Cover

(Degree-1)

If $\exists v \in V$ such that $d_G(v) = 1$, then set $G \leftarrow G - N_G[v]$ and $k \leftarrow k - 1$.

Simplification rules for VERTEX COVER

(Degree-1)

If $\exists v \in V$ such that $d_G(v) = 1$, then set $G \leftarrow G - N_G[v]$ and $k \leftarrow k - 1$.

Lemma 1

(Degree-1) is sound.

Proof.

Let u be the neighbor of v in G. Thus, $N_G[v] = \{u, v\}$.
If S is a vertex cover of G of size at most k, then $S \setminus \{u, v\}$ is a vertex cover of $G - N_G[v]$ of size at most $k - 1$, because $u \in S$ or $v \in S$.
If S' is a vertex cover of $G - N_G[v]$ of size at most $k - 1$, then $S' \cup \{u\}$ is a vertex cover of G of size at most k, since all edges that are in G but not in $G - N_G[v]$ are incident to v. $\square$

Simplification rules for Vertex Cover

(Large Degree)

If $\exists v \in V$ such that $d_G(v) > k$, then set $G \leftarrow G - v$ and $k \leftarrow k - 1$.

Simplification rules for Vertex Cover

(Large Degree)

If $\exists v \in V$ such that $d_G(v) > k$, then set $G \leftarrow G - v$ and $k \leftarrow k - 1$.

Lemma 1

(Large Degree) is sound.

Proof.

Let S be a vertex cover of G of size at most k. If $v \notin S$, then $N_G(v) \subseteq S$, contradicting that $|S| \leq k$. □

Simplification rules for Vertex Cover

(Number of Edges)

If $d_G(v) \leq k$ for each $v \in V$ and $|E| > k^2$ then return No

Simplification rules for Vertex Cover

(Number of Edges)

If $d_G(v) \leq k$ for each $v \in V$ and $|E| > k^2$ then return No

Lemma 1

(Number of Edges) is sound.

Proof.

Assume $d_G(v) \leq k$ for each $v \in V$ and $|E| > k^2$.
Suppose $S \subseteq V$, $|S| \leq k$, is a vertex cover of G.
We have that S covers at most k^2 edges.
However, $|E| \geq k^2 + 1$.
Thus, S is not a vertex cover of G. □

Outline

Preprocessing algorithm for VERTEX COVER

VC-preprocess
Input: A graph G and an integer k.
Output: A graph G' and an integer k' such that G has a vertex cover of size at most k if and only if G' has a vertex cover of size at most k'.

$G' \leftarrow G$
$k' \leftarrow k$
repeat
    Execute simplification rules (Degree-0), (Degree-1), (Large Degree), and (Number of Edges) for (G', k')
until *no simplification rule applies*
return (G', k')

Effectiveness of preprocessing algorithms

- How effective is VC-preprocess?
- We would like to study preprocessing algorithms mathematically and quantify their effectiveness.

First try

- Say that a preprocessing algorithm for a problem Π is nice if it runs in polynomial time and for each instance for Π, it returns an instance for Π that is strictly smaller.

First try

- Say that a preprocessing algorithm for a problem Π is nice if it runs in polynomial time and for each instance for Π, it returns an instance for Π that is strictly smaller.
- $\rightarrow$ executing it a linear number of times reduces the instance to a single bit
- $\rightarrow$ such an algorithm would solve Π in polynomial time
- For NP-hard problems this is not possible unless P $=$ NP
- We need a different measure of effectiveness

Measuring the effectiveness of preprocessing algorithms

- We will measure the effectiveness in terms of the parameter
- How large is the resulting instance in terms of the parameter?

Effectiveness of VC-preprocess

Lemma 2

For any instance (G, k) for Vertex Cover*, VC-preprocess produces an equivalent instance (G', k') of size $O(k^2)$.*

Proof.

Since all simplification rules are sound, $(G = (V, E), k)$ and $(G' = (V', E'), k')$ are equivalent.
By (Number of Edges), $|E'| \leq (k')^2 \leq k^2$.
By (Degree-0) and (Degree-1), each vertex in V' has degree at least 2 in G'.
Since $\sum_{v \in V'} d_{G'}(v) = 2|E'| \leq 2k^2$, this implies that $|V'| \leq k^2$.
Thus, $|V'| + |E'| \subseteq O(k^2)$. □

Outline

Kernelization: definition

Definition 3

A kernelization for a parameterized problem Π is a **polynomial time** algorithm, which, for any instance I of Π with parameter k, produces an **equivalent** instance I' of Π with parameter k' such that $|I'| \leq f(k)$ and $k' \leq f(k)$ for a computable function f.
We refer to the function f as the size of the kernel.

Note: We do not formally require that $k' \leq k$, but this will be the case for many kernelizations.

VC-preprocess is a quadratic kernelization

Theorem 4

VC-preprocess is a $O(k^2)$ kernelization for VERTEX COVER.

Can we obtain a kernel with fewer vertices?

Outline

Integer Linear Program for Vertex Cover

The Vertex Cover problem can be written as an Integer Linear Program (ILP).
For an instance $(G = (V, E), k)$ for Vertex Cover with $V = \{v_1, \ldots, v_n\}$, create a variable x_i for each vertex v_i, $1 \leq i \leq n$.
Let $X = \{x_1, \ldots, x_n\}$.

$$\mathrm{ILP}_{\mathrm{VC}}(G) = \boxed{\begin{array}{rl} \text{Minimize} \displaystyle\sum_{i=1}^{n} x_i & \\ x_i + x_j \geq 1 & \text{for each } \{v_i, v_j\} \in E \\ x_i \in \{0, 1\} & \text{for each } i \in \{1, \ldots, n\} \end{array}}$$

Then, (G, k) is a Yes-instance iff the objective value of $\mathrm{ILP}_{\mathrm{VC}}(G)$ is at most k.

LP relaxation for Vertex Cover

$$\text{LP}_{\text{VC}}(G) = \boxed{\begin{aligned} &\text{Minimize} \sum_{i=1}^{n} x_i & \\ &x_i + x_j \geq 1 & \text{for each } \{v_i, v_j\} \in E \\ &x_i \geq 0 & \text{for each } i \in \{1, \dots, n\} \end{aligned}}$$

Note: the value of an optimal solution for the Linear Program $\text{LP}_{\text{VC}}(G)$ is at most the value of an optimal solution for $\text{ILP}_{\text{VC}}(G)$

Properties of LP optimal solution

- Let $\alpha : X \to \mathbb{R}_{\geq 0}$ be an optimal solution for $\mathsf{LP}_{\mathsf{VC}}(G)$. Let

$$\begin{aligned} V_- &= \{v_i : \alpha(x_i) < 1/2\} \\ V_{1/2} &= \{v_i : \alpha(x_i) = 1/2\} \\ V_+ &= \{v_i : \alpha(x_i) > 1/2\} \end{aligned}$$

Properties of LP optimal solution

- Let $\alpha : X \to \mathbb{R}_{\geq 0}$ be an optimal solution for $\text{LP}_{\text{VC}}(G)$. Let

$$\begin{aligned} V_- &= \{v_i : \alpha(x_i) < 1/2\} \\ V_{1/2} &= \{v_i : \alpha(x_i) = 1/2\} \\ V_+ &= \{v_i : \alpha(x_i) > 1/2\} \end{aligned}$$

Lemma 5

For each $i, 1 \leq i \leq n$, we have that $\alpha(x_i) \leq 1$.

Lemma 6

V_- is an independent set.

Lemma 7

$N_G(V_-) = V_+$.

Properties of LP optimal solution II

Lemma 8

For each $S \subseteq V_+$ *we have that* $|S| \leq |N_G(S) \cap V_-|$.

Proof.

For the sake of contradiction, suppose there is a set $S \subseteq V_+$ such that $|S| > |N_G(S) \cap V_-|$.

Let $\epsilon = \min_{v_i \in S}\{\alpha(x_i) - 1/2\}$ and $\alpha' : X \to \mathbb{R}_{\geq 0}$ s.t.

$$\alpha'(x_i) = \begin{cases} \alpha(x_i) & \text{if } v_i \notin S \cup (N_G(S) \cap V_-) \\ \alpha(x_i) - \epsilon & \text{if } v_i \in S \\ \alpha(x_i) + \epsilon & \text{if } v_i \in N_G(S) \cap V_- \end{cases}$$

Note that α' is an improved solution for $\mathsf{LP}_{\mathsf{VC}}(G)$, contradicting that α is optimal. □

Properties of LP optimal solution III

Theorem 9 (Hall's marriage theorem)

A bipartite graph $G = (V \uplus U, E)$ *has a matching saturating* $S \subseteq V$

$$\Leftrightarrow$$

for every subset $W \subseteq S$ *we have* $|W| \leq |N_G(W)|$. [1]

[1] A matching M in a graph G is a set of edges such that no two edges in M have a common endpoint. A matching saturates a set of vertices S if each vertex in S is an end point of an edge in M.

Properties of LP optimal solution III

Theorem 9 (Hall's marriage theorem)

A bipartite graph $G = (V \uplus U, E)$ has a matching saturating $S \subseteq V$

$$\Leftrightarrow$$

for every subset $W \subseteq S$ we have $|W| \leq |N_G(W)|$. [1]

Consider the bipartite graph $B = (V_- \uplus V_+, \{\{u,v\} \in E : u \in V_-, v \in V_+\})$.

Lemma 10

There exists a matching M in B of size $|V_+|$.

Proof.

The lemma follows from the previous lemma and Hall's marriage theorem. □

[1] A matching M in a graph G is a set of edges such that no two edges in M have a common endpoint. A matching saturates a set of vertices S if each vertex in S is an end point of an edge in M.

Crown Decomposition: Definition

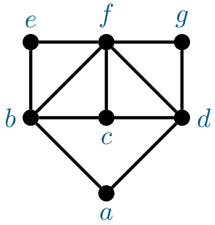
Definition 11 (Crown Decomposition)

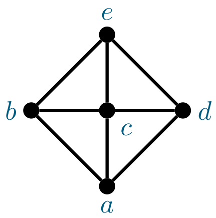
A crown decomposition (C, H, B) of a graph $G = (V, E)$ is a partition of V into sets C, H, and B such that

- the crown C is a non-empty independent set,
- the head $H = N_G(C)$,
- the body $B = V \setminus (C \cup H)$, and
- there is a matching of size $|H|$ in $G[H \cup C]$.

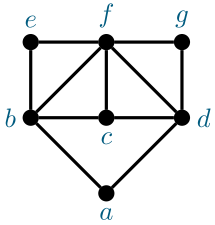
By the previous lemmas, we obtain a crown decomposition $(V_-, V_+, V_{1/2})$ of G if $V_- \neq \emptyset$.

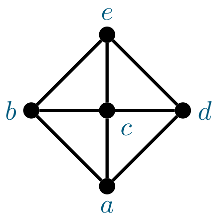
Crown Decomposition: Examples





Crown Decomposition: Examples





crown decomposition
$(\{a, e, g\}, \{b, d, f\}, \{c\})$

has no crown decomposition

Using the crown decomposition

Lemma 12

Suppose that $G = (V, E)$ has a crown decomposition (C, H, B). Then,

$$\mathit{vc}(G) \le k \quad \Leftrightarrow \quad \mathit{vc}(G[B]) \le k - |H|,$$

where $\mathit{vc}(G)$ denotes the size of the smallest vertex cover of G.

Using the crown decomposition

Lemma 12

Suppose that $G = (V, E)$ *has a crown decomposition* (C, H, B)*. Then,*

$$\mathit{vc}(G) \leq k \quad \Leftrightarrow \quad \mathit{vc}(G[B]) \leq k - |H|,$$

where $\mathit{vc}(G)$ *denotes the size of the smallest vertex cover of* G*.*

Proof.

$(\Rightarrow)$: Let S be a vertex cover of G with $|S| \leq k$. Since S contains at least one vertex for each edge of a matching, $|S \cap (C \cup H)| \geq |H|$. Therefore, $S \cap B$ is a vertex cover for $G[B]$ of size at most $k - |H|$.
$(\Leftarrow)$: Let S be a vertex cover of $G[B]$ with $|S| \leq k - |H|$. Then, $S \cup H$ is a vertex cover of G of size at most k, since each edge that is in G but not in G' is incident to a vertex in H. □

Nemhauser-Trotter

Corollary 13 ([Nemhauser, Trotter, 1974])

There exists a smallest vertex cover S of G such that $S \cap V_- = \emptyset$ and $V_+ \subseteq S$.

Crown reduction

(Crown Reduction)

If solving $\mathrm{LP}_{VC}(G)$ gives an optimal solution with $V_- \neq \emptyset$, then return $(G - (V_- \cup V_+), k - |V_+|)$.

Crown reduction

(Crown Reduction)

If solving LP$_{VC}(G)$ gives an optimal solution with $V_- \neq \emptyset$, then return $(G - (V_- \cup V_+), k - |V_+|)$.

(Number of Vertices)

If solving LP$_{VC}(G)$ gives an optimal solution with $V_- = \emptyset$ and $|V| > 2k$, then return No.

Crown reduction

(Crown Reduction)

If solving $\text{LP}_{VC}(G)$ gives an optimal solution with $V_- \neq \emptyset$, then return $(G - (V_- \cup V_+), k - |V_+|)$.

(Number of Vertices)

If solving $\text{LP}_{VC}(G)$ gives an optimal solution with $V_- = \emptyset$ and $|V| > 2k$, then return No.

Lemma 14

(Crown Reduction) and (Number of Vertices) are sound.

Proof.

(Crown Reduction) is sound by previous Lemmas.
Let α be an optimal solution for $\text{LP}_{VC}(G)$ and suppose $V_- = \emptyset$. The value of this solution is at least $|V|/2$. Thus, the value of an optimal solution for $\text{ILP}_{\text{VC}}(G)$ is at least $|V|/2$. Since G has no vertex cover of size less than $|V|/2$, we have a No-instance if $k < |V|/2$. □

Linear vertex-kernel for Vertex Cover

Theorem 15

Vertex Cover *has a kernel with* $2k$ *vertices and* $O(k^2)$ *edges.*

This is the smallest known kernel for Vertex Cover.
See http://fpt.wikidot.com/fpt-races for the current smallest kernels for various problems.

Outline

1

Vertex Cover1section.1 1.1Simplification rules1subsection.1.1 1.2Preprocessing algorithm2subsection.1.2 2Kernelization algorithms3section.2 3A smaller kernel for VERTEX COVER3section.3 4More on Crown Decompositions5section.4 5Kernels and Fixed-parameter tractability6section.5 6Further Reading6section.6

Crown Lemma

Lemma 16 (Crown Lemma)

Let $G = (V, E)$ be a graph without isolated vertices and with $|V| \geq 3k + 1$. There is a polynomial time algorithm that either

- *finds a matching of size $k + 1$ in G, or*
- *finds a crown decomposition of G.*

Crown Lemma

Lemma 16 (Crown Lemma)

Let $G = (V, E)$ *be a graph without isolated vertices and with* $|V| \geq 3k + 1$*. There is a polynomial time algorithm that either*

- *finds a matching of size* $k + 1$ *in* G*, or*
- *finds a crown decomposition of* G*.*

To prove the lemma, we need Kőnig's Theorem

Theorem 17 ([Kőnig, 1916])

In every bipartite graph the size of a maximum matching is equal to the size of a minimum vertex cover.

Crown Lemma

Lemma 16 (Crown Lemma)

Let $G = (V, E)$ be a graph without isolated vertices and with $|V| \geq 3k + 1$. There is a polynomial time algorithm that either

- *finds a matching of size $k + 1$ in G, or*
- *finds a crown decomposition of G.*

Proof.

Compute a maximum matching M of G. If $|M| \geq k + 1$, we are done.

□

Crown Lemma

Lemma 16 (Crown Lemma)

Let $G = (V, E)$ be a graph without isolated vertices and with $|V| \geq 3k + 1$. There is a polynomial time algorithm that either

- *finds a matching of size $k + 1$ in G, or*
- *finds a crown decomposition of G.*

Proof.

Compute a maximum matching M of G. If $|M| \geq k + 1$, we are done. Note that $I := V \setminus V(M)$ is an independent set with $\geq k + 1$ vertices.

$\square$

Crown Lemma

Lemma 16 (Crown Lemma)

Let $G = (V, E)$ be a graph without isolated vertices and with $|V| \geq 3k + 1$. There is a polynomial time algorithm that either

- *finds a matching of size $k + 1$ in G, or*
- *finds a crown decomposition of G.*

Proof.

Compute a maximum matching M of G. If $|M| \geq k + 1$, we are done.
Note that $I := V \setminus V(M)$ is an independent set with $\geq k + 1$ vertices.
Consider the bipartite graph B formed by edges with one endpoint in $V(M)$ and the other in I.

$\square$

Crown Lemma

Lemma 16 (Crown Lemma)

Let $G = (V, E)$ be a graph without isolated vertices and with $|V| \geq 3k + 1$. There is a polynomial time algorithm that either

- *finds a matching of size $k + 1$ in G, or*
- *finds a crown decomposition of G.*

Proof.

Compute a maximum matching M of G. If $|M| \geq k + 1$, we are done.
Note that $I := V \setminus V(M)$ is an independent set with $\geq k + 1$ vertices.
Consider the bipartite graph B formed by edges with one endpoint in $V(M)$ and the other in I.
Compute a minimum vertex cover X and a maximum matching M' of B.

□

Crown Lemma

Lemma 16 (Crown Lemma)

Let $G = (V, E)$ be a graph without isolated vertices and with $|V| \geq 3k + 1$. There is a polynomial time algorithm that either

- *finds a matching of size $k + 1$ in G, or*
- *finds a crown decomposition of G.*

Proof.

Compute a maximum matching M of G. If $|M| \geq k + 1$, we are done.
Note that $I := V \setminus V(M)$ is an independent set with $\geq k + 1$ vertices.
Consider the bipartite graph B formed by edges with one endpoint in $V(M)$ and the other in I.
Compute a minimum vertex cover X and a maximum matching M' of B.
We know: $|X| = |M'| \leq |M| \leq k$. Hence, $X \cap V(M) \neq \emptyset$.

□

Crown Lemma

Lemma 16 (Crown Lemma)

Let $G = (V, E)$ be a graph without isolated vertices and with $|V| \geq 3k + 1$. There is a polynomial time algorithm that either

- *finds a matching of size $k + 1$ in G, or*
- *finds a crown decomposition of G.*

Proof.

Compute a maximum matching M of G. If $|M| \geq k + 1$, we are done.
Note that $I := V \setminus V(M)$ is an independent set with $\geq k + 1$ vertices.
Consider the bipartite graph B formed by edges with one endpoint in $V(M)$ and the other in I.
Compute a minimum vertex cover X and a maximum matching M' of B.
We know: $|X| = |M'| \leq |M| \leq k$. Hence, $X \cap V(M) \neq \emptyset$.
Let $M^* = \{e \in M' : e \cap (X \cap V(M)) \neq \emptyset\}$.

□

Crown Lemma

Lemma 16 (Crown Lemma)

Let $G = (V, E)$ be a graph without isolated vertices and with $|V| \geq 3k + 1$. There is a polynomial time algorithm that either

- *finds a matching of size $k + 1$ in G, or*
- *finds a crown decomposition of G.*

Proof.

Compute a maximum matching M of G. If $|M| \geq k + 1$, we are done.
Note that $I := V \setminus V(M)$ is an independent set with $\geq k + 1$ vertices.
Consider the bipartite graph B formed by edges with one endpoint in $V(M)$ and the other in I.
Compute a minimum vertex cover X and a maximum matching M' of B.
We know: $|X| = |M'| \leq |M| \leq k$. Hence, $X \cap V(M) \neq \emptyset$.
Let $M^* = \{e \in M' : e \cap (X \cap V(M)) \neq \emptyset\}$.
We obtain a crown decomposition with crown $C = V(M^*) \cap I$ and head $H = X \cap V(M) = X \cap V(M^*)$. □

Exercise

A k-coloring of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, ..., k\}$ such that $f(u) \neq f(v)$ if $uv \in E$.

Saving Colors	
Input:	Graph G, integer k
Parameter:	k
Question:	Does G have a $(n-k)$-coloring?

Design a kernel for Saving Colors with $O(k)$ vertices.

Outline

Kernels and Fixed-parameter tractability

Theorem 17

Let Π *be a decidable parameterized problem.*
Π *has a kernelization algorithm* $\Leftrightarrow$ Π *is* FPT.

Kernels and Fixed-parameter tractability

Theorem 17

Let Π be a decidable parameterized problem.
Π has a kernelization algorithm $\Leftrightarrow$ Π is FPT.

Proof.

$(\Rightarrow)$: An FPT algorithm is obtained by first running the kernelization, and then any brute-force algorithm on the resulting instance.
$(\Leftarrow)$: Let A be an FPT algorithm for Π with running time $O(f(k)n^c)$.
If $f(k) < n$, then A has running time $O(n^{c+1})$. In this case, the kernelization algorithm runs A and returns a trivial YES- or NO-instance depending on the answer of A.
Otherwise, $f(k) \geq n$. In this case, the kernelization algorithm outputs the input instance. $\square$

After computing a kernel ...

- ... we can use any algorithm to compute an actual solution.
- Brute-force, faster exponential-time algorithms, parameterized algorithms, often also approximation algorithms

Kernels

- A parameterized problem may not have a kernelization algorithm
  - Example, COLORING[2] parameterized by k has no kernelization algorithm unless $\mathsf{P} = \mathsf{NP}$.
  - A kernelization would lead to a polynomial time algorithm for the NP-complete 3-COLORING problem
- Kernelization algorithms lead to FPT algorithms ...
- ... FPT algorithms lead to kernels

[2] Can one color the vertices of an input graph G with k colors such that no two adjacent vertices receive the same color?

Outline

Further Reading

- Chapter 2, *Kernelization* in
  Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, MichałPilipczuk, and Saket Saurabh. Parameterized Algorithms. Springer, 2015.
- Chapter 4, *Kernelization* in
  Rodney G. Downey and Michael R. Fellows. Fundamentals of Parameterized Complexity. Springer, 2013.
- Chapter 7, *Data Reduction and Problem Kernels* in
  Rolf Niedermeier. Invitation to Fixed Parameter Algorithms. Oxford University Press, 2006.
- Chapter 9, *Kernelization and Linear Programming Techniques* in
  Jörg Flum and Martin Grohe. Parameterized Complexity Theory. Springer, 2006.