Exercise sheet 4a – Solutions and Hints
COMP6741: Parameterized and Exact Computation
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Exercise 1.  A Boolean formula in Conjunctive Normal Form (CNF) is a conjunction (AND) of disjunctions (OR) of literals (a Boolean variable or its negation). A HORN formula is a CNF formula where each clause contains at most one positive literal. For a CNF formula $F$ and an assignment $\tau: S \to \{0, 1\}$ to a subset $S$ of its variables, the formula $F[\tau]$ is obtained from $F$ by removing each clause that contains a literal that evaluates to 1 under $S$, and removing all literals that evaluate to 0 from the remaining clauses.

HORN-Backdoor Detection
Input: A CNF formula $F$ and an integer $k$.
Parameter: $k$
Question: Is there a subset $S$ of the variables of $F$ with $|S| \leq k$ such that for each assignment $\tau: S \to \{0, 1\}$, the formula $F[\tau]$ is a HORN formula?

Example: $(\neg a \lor b \lor c) \land (b \lor \neg c \lor \neg d) \land (a \lor b \lor \neg c) \land (\neg b \lor c \lor \neg e)$ with $k = 1$ is a Yes-instance, certified by $S = \{b\}$.

- Show that HORN-Backdoor Detection is FPT using the fact that VERTEX COVER is FPT.

Hint.
- Show the following: if two distinct positive literals occur in a same clause, then a HORN-backdoor must contain at least one of the corresponding variables.
- Construct a parameterized reduction to VERTEX COVER based on these pairwise conflicts.

Exercise 2.  Show that Weighted Circuit Satisfiability $\in X\text{P}$.

Hint.
- There are $n^k$ assignments of weight $k$, where $n$ is the number of input gates.

Exercise 3.  Recall that a $k$-coloring of a graph $G = (V, E)$ is a function $f: V \to \{1, 2, \ldots, k\}$ assigning colors to $V$ such that no two adjacent vertices receive the same color.

Multicolor Clique
Input: A graph $G = (V, E)$, an integer $k$, and a $k$-coloring of $G$
Parameter: $k$
Question: Does $G$ have a clique of size $k$?

- Show that Multicolor Clique is W[1]-hard.

Hint: Reduce from CLIQUE, and create $k$ copies of $V$, each one being an independent set in $G'$. Add edges to enforce constraints that a clique of size $k$ in $G'$ corresponds to a clique of size $k$ in $G$, and vice-versa.

Solution. The proof is by a parameterized reduction from CLIQUE.

Construction. Let $(G = (V, E), k)$ be an instance for CLIQUE. We construct an instance $(G' = (V', E'), k', f)$ for Multicolor Clique as follows. For each $v \in V$, create $k$ vertices $v(1), \ldots, v(k)$ and add them to $V'$. For every
pair \(u(i), v(j) \in V'\) with \(i \neq j\), add \(u(i)v(j)\) to \(E'\) if and only if \(uv \in E\). Set \(k' := k\). Set \(f(v(i)) = i\) for each \(v \in V\) and \(i \in \{1, \ldots, k\}\).

**Equivalence.** \(G\) has a clique of size \(k\) if and only if \(G'\) has a clique of size \(k\).

\((\Rightarrow):\) Let \(S = \{s_1, \ldots, s_k\}\) be a clique in \(G\). Then \(S' = \{s_1(1), s_2(2), \ldots, s_k(k)\}\) is a clique in \(G'\) since \(s_is_j \in E\) implies \(s_i(s_j) \in E'\) in our construction.

\((\Leftarrow):\) Let \(S'\) be a clique of size \(k\) in \(G'\). Since for each \(i \in \{1, \ldots, k\}\), \(\{v_i : v \in V\}\) is an independent set in \(G'\), \(S'\) contains exactly one vertex from each color class of \(f\). Denote \(S' = \{s'_1(1), \ldots, s'_k(k)\}\). Then, \(S = \{s_1, \ldots, s_k\}\) is a clique in \(G\).

**Parameter.** \(k' \leq k\).

**Running time.** The construction can clearly be done in FPT time, and even in polynomial time.

**Exercise 4.** A set system \(S\) is a pair \((V, H)\), where \(V\) is a finite set of elements and \(H\) is a set of subsets of \(V\). A set cover of a set system \(S = (V, H)\) is a subset \(X\) of \(H\) such that each element of \(V\) is contained in at least one of the sets in \(X\), i.e., \(\bigcup_{Y \in X} Y = V\).

<table>
<thead>
<tr>
<th><strong>Set Cover</strong></th>
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<tbody>
<tr>
<td>Input: A set system (S = (V, H)) and an integer (k)</td>
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<tr>
<td>Parameter: (k)</td>
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<tr>
<td>Question: Does (S) have a set cover of cardinality at most (k)?</td>
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- Show that **Set Cover** is W[2]-hard.

**Hint.** Reduce from **Dominating Set**:

- add an element for each vertex and
- add a set for each vertex, containing all the vertices in its closed neighborhood.

**Exercise 5.** A hitting set of a set system \(S = (V, H)\) is a subset \(X\) of \(V\) such that \(X\) contains at least one element of each set in \(H\), i.e., \(X \cap Y \neq \emptyset\) for each \(Y \in H\).

<table>
<thead>
<tr>
<th><strong>Hitting Set</strong></th>
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<tbody>
<tr>
<td>Input: A set system (S = (V, H)) and an integer (k)</td>
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<td>Parameter: (k)</td>
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<tr>
<td>Question: Does (S) have a hitting set of size at most (k)?</td>
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- Show that **Hitting Set** is W[2]-hard.

**Hint:** Exploit a duality between sets and elements in set covers and hitting sets.

**Solution sketch.** Reduce from **Set Cover**. Let \((S = (V, H), k)\) be an instance for **Set Cover**. Construct an instance \((S' = (V', H'), k)\) for **Hitting Set**:

- \(V' := H\)
- \(H' := \{\{h \in H : v \in h\} : v \in V\}\)