5. Basics of Parameterized Complexity
COMP6741: Parameterized and Exact Computation

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Semester 2, 2015
Outline

1. Introduction
   - Vertex Cover
   - Coloring
   - Clique
   - $\Delta$-Clique

2. Basic Definitions

3. Further Reading
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- Vertex Cover
- Coloring
- Clique
- \(\Delta\)-Clique

2 Basic Definitions

3 Further Reading
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   - Vertex Cover
   - Coloring
   - Clique
   - $\Delta$-Clique

2. Basic Definitions

3. Further Reading
A vertex cover in a graph $G = (V, E)$ is a subset of its vertices $S \subseteq V$ such that every edge of $G$ has at least one endpoint in $S$.

**Vertex Cover**

Input: A graph $G = (V, E)$ and an integer $k$

Parameter: $k$

Question: Does $G$ have a vertex cover of size $k$?
Algorithms for Vertex Cover

- brute-force: $O^*(2^n)$
- brute-force: $O^*(n^k)$
- vc1: $O^*(2^k)$ (cf. Lecture 1)
- vc2: $O^*(1.4656^k)$ (cf. Lecture 1)
- fastest known: $O(1.2738^k + k \cdot n)$ [Chen, Kanj, Xia, 2010]
Running times in practice

$n = 1000$ vertices,  
$k = 20$ parameter

<table>
<thead>
<tr>
<th>Theoretical</th>
<th>Running Time Nb of Instructions</th>
<th>Real</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^n$</td>
<td>$1.07 \cdot 10^{301}$</td>
<td>$4.941 \cdot 10^{282}$ years</td>
</tr>
<tr>
<td>$n^k$</td>
<td>$10^{60}$</td>
<td>$4.611 \cdot 10^{41}$ years</td>
</tr>
<tr>
<td>$2^k \cdot n$</td>
<td>$1.05 \cdot 10^9$</td>
<td>$15.26$ milliseconds</td>
</tr>
<tr>
<td>$1.4656^k \cdot n$</td>
<td>$2.10 \cdot 10^6$</td>
<td>$0.31$ milliseconds</td>
</tr>
<tr>
<td>$1.2738^k + k \cdot n$</td>
<td>$2.02 \cdot 10^4$</td>
<td>$0.0003$ milliseconds</td>
</tr>
</tbody>
</table>

Notes:
– We assume that $2^{36}$ instructions are carried out per second.
– The Big Bang happened roughly $13.5 \cdot 10^9$ years ago.
Goal of Parameterized Complexity

Confine the combinatorial explosion to a parameter $k$.

(1) Which problem–parameter combinations are fixed-parameter tractable (FPT)? In other words, for which problem–parameter combinations are there algorithms with running times of the form

$$f(k) \cdot n^{O(1)},$$

where the $f$ is a computable function independent of the input size $n$?

(2) How small can we make the $f(k)$?
Examples of Parameters

A Parameterized Problem

Input: an instance of the problem
Parameter: a parameter
Question: a Yes–No question about the instance and the parameter

A parameter can be

- solution size
- input size (trivial parameterization)
- related to the structure of the input (maximum degree, treewidth, branchwidth, genus, ...)
- combinations of parameters
- etc.
Outline

1 Introduction
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   - Coloring
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   - $\Delta$-Clique

2 Basic Definitions

3 Further Reading
A $k$-coloring of a graph $G = (V, E)$ is a function $f : V \rightarrow \{1, 2, ..., k\}$ assigning colors to $V$ such that no two adjacent vertices receive the same color.

**COLORING**

Input: Graph $G$, integer $k$
Parameter: $k$
Question: Does $G$ have a $k$-coloring?

Brute-force: $O^*(k^n)$, where $n = |V(G)|$.
Inclusion-Exclusion: $O^*(2^n)$.
FPT?
Known: \textsc{Coloring} is \textbf{NP}-complete when $k = 3$

Suppose there was a $O^*(f(k))$-time algorithm for \textsc{Coloring}

- Then, 3-\textsc{Coloring} can be solved in $O^*(f(3)) \leq O^*(1)$ time
- Therefore, $P = \text{NP}$

Therefore, \textsc{Coloring} is not \textbf{FPT} unless $P = \text{NP}$
Outline

1. Introduction
   - Vertex Cover
   - Coloring
   - Clique
   - \( \Delta \)-Clique

2. Basic Definitions

3. Further Reading
A **clique** in a graph $G = (V, E)$ is a subset of its vertices $S \subseteq V$ such that every two vertices from $S$ are adjacent in $G$.

**CLIQUE**

Input: Graph $G = (V, E)$, integer $k$
Parameter: $k$
Question: Does $G$ have a clique of size $k$?

Is **CLIQUE** NP-complete when $k$ is a fixed constant? Is it FPT?
Algorithm for Clique

For each subset $S \subseteq V$ of size $k$, check whether all vertices of $S$ are adjacent

- Running time: $O^* \left( \binom{n}{k} \right) \subseteq O^* \left( n^k \right)$
- When $k \in O(1)$, this is polynomial
- But: we do not currently know an FPT algorithm for Clique
- Since Clique is W[1]-hard, we believe it is not FPT. (See lecture on W-hardness.)
Outline

1 Introduction
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   - Coloring
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2 Basic Definitions

3 Further Reading
A different parameter for Clique

\[ \Delta\text{-Clique} \]

**Input:** Graph \( G = (V, E) \), integer \( k \)

**Parameter:** \( \Delta(G) \), i.e., the maximum degree of \( G \)

**Question:** Does \( G \) have a clique of size \( k \)?

Is \( \Delta\text{-Clique} \) FPT?
Algorithm for $\Delta$-Clique

- If $k = 0$, answer $\text{YES}$.
- If $k > \Delta + 1$, answer $\text{NO}$.
- Otherwise,
  - // A clique of size $k$ contains at least one vertex $v$. We try all possibilities for $v$. 

Running time: $O^{\ast}((\Delta + 1)^k) \subseteq O^{\ast}((\Delta + 1)^{\Delta})$. (FPT for parameter $\Delta$)

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Semester 2, 2015
Algorithm for $\Delta$-Clique

- If $k = 0$, answer Yes.
- If $k > \Delta + 1$, answer No.
- Otherwise,
  - // A clique of size $k$ contains at least one vertex $v$. We try all possibilities for $v$.
  - // For each $v \in V$, we will check whether $G$ has a clique of size $k$ containing $v$.
  - // Note that for a clique $S$ containing $v$, we have $S \subseteq N_G[v]$.
  - For each $v \in V$, check for each vertex subset $S \subseteq N_G[v]$ of size $k$ whether $S$ is a clique in $G$. 

Running time: $O^*((\Delta + 1)^k) \subseteq O^*((\Delta + 1)^\Delta)$ (FPT for parameter $\Delta$)
Algorithm for $\Delta$-Clique

- If $k = 0$, answer **Yes**.
- If $k > \Delta + 1$, answer **No**.
- Otherwise,
  - // A clique of size $k$ contains at least one vertex $v$. We try all possibilities for $v$.
  - // For each $v \in V$, we will check whether $G$ has a clique of size $k$ containing $v$.
  - // Note that for a clique $S$ containing $v$, we have $S \subseteq N_G[v]$.
  - For each $v \in V$, check for each vertex subset $S \subseteq N_G[v]$ of size $k$ whether $S$ is a clique in $G$.

- Running time: $O^*((\Delta + 1)^k) \subseteq O^*((\Delta + 1)^{\Delta})$. (**FPT** for parameter $\Delta$)
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1 Introduction
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2 Basic Definitions

3 Further Reading
Main Parameterized Complexity Classes

\( n \): instance size
\( k \): parameter

\( P \): class of problems that can be solved in \( n^{O(1)} \) time
\( \text{FPT} \): class of parameterized problems that can be solved in \( f(k) \cdot n^{O(1)} \) time
\( W[\cdot] \): parameterized intractability classes
\( \text{XP} \): class of parameterized problems that can be solved in \( f(k) \cdot n^{g(k)} \) time
  (“polynomial when \( k \) is a constant”)

\[
P \subseteq \text{FPT} \subseteq W[1] \subseteq W[2] \cdots \subseteq W[P] \subseteq \text{XP}
\]

\textbf{Known:} If \( \text{FPT} = W[1] \), then the Exponential Time Hypothesis fails, i.e. 3-\text{SAT} can be solved in \( 2^{o(n)} \) time, where \( n \) is the number of variables.

\textbf{Note:} We assume that \( f \) is \textit{computable} and \textit{non-decreasing}.
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2 Basic Definitions

3 Further Reading
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