Exercise sheet 1 – Solutions

COMP6741: Parameterized and Exact Computation

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Exercise 1. Arrange the following functions in increasing order of growth: $n^{\log n}$; $(\log n)^n$; 2^n; 2^{2^n}; 2^{n^2}; $n!$; 1.01^n; 50^n; $2^{n/2}$; $2^{\sqrt{n}}$.

Solution. By increasing order of growth: $n^{\log n}$; $2^{\sqrt{n}}$; 1.01^n; $2^{n/2}$; 2^n; 50^n; $(\log n)^n$; $n!$; 2^{n^2}; 2^{2^n}.

Exercise 2. Show that Vertex Cover can be solved in polynomial time for graphs of maximum degree at most 2.

Solution. First, we make an observation about the structure of such graphs.

Observation 1. *A graph of maximum degree at most 2 is a disjoint union of paths and cycles* [1].

Denote by $vc_{opt}(G)$ the vertex cover number of G, i.e., the size of a smallest vertex cover of G. By the following observation, the vertex cover number of G equals the sum of the vertex cover numbers of the connected components of G.

Observation 2. *A (smallest) vertex cover of a graph is the union of (smallest) vertex covers of each of its connected components.*

Now, it suffices to optimally solve Vertex Cover on these two types of graphs.

Lemma 3. *For a path P_k on $k \geq 1$ vertices, $vc_{opt}(P_k) = \lceil (k-1)/2 \rceil$.*

Proof. The proof is by induction on k.

For the base cases $k = 1$ and $k = 2$, note that an edgeless graph has an empty vertex cover, and a graph with a single edge has an optimal vertex cover of size 1. Therefore, $vc_{opt}(P_1) = \lceil (1-1)/2 \rceil = 0$ and $vc_{opt}(P_2) = \lceil (2-1)/2 \rceil = 1$, as required.

To prove that the lemma holds for $k \geq 3$, assume it holds for all k' with $1 \leq k' < k$. Denote the sequence of vertices of the path P_k by $(v_1, v_2, \ldots, v_k)$. To cover the edge $v_{k-1}v_k$, a vertex cover C needs to include v_{k-1} or v_k (or both). If $v_k \in C$, then $C' = (C \setminus \{v_k\}) \cup \{v_{k-1}\}$ is a vertex cover as well, and $|C'| \leq |C|$. We conclude that there is a smallest vertex cover containing v_{k-1}. The remaining vertices of the vertex cover need to cover the edges of the path $P_{k-2} = (v_1, v_2, \ldots, v_{k-2})$. Therefore,

$$vc_{opt}(P_k) = 1 + vc_{opt}(P_{k-2}) = 1 + \lceil (k-3)/2 \rceil = \lceil (k-1)/2 \rceil.$$

This concludes the proof of the lemma. □

Lemma 4. *For a cycle C_k on $k \geq 3$ vertices, $vc_{opt}(C_k) = \lceil k/2 \rceil$.*

Proof. Since $C_k = (v_1, v_2, \ldots, v_k, v_1)$ has edges, its smallest vertex cover contains at least one vertex. By symmetry, there is a smallest vertex cover containing the vertex v_k. The remaining vertices of the vertex cover need to cover the vertices of the path $P_{k-1} = (v_1, v_2, \ldots, v_{k-1})$. Thus,

$$vc_{opt}(C_k) = 1 + vc_{opt}(P_{k-1}) = 1 + \lceil (k-2)/2 \rceil = \lceil k/2 \rceil.$$

This concludes the proof of the lemma. □

[1] see the Glossary if these terms are unclear

To solve Vertex Cover on a graph of maximum degree at most 2, we compute its connected components (for example, by breadth-first search). For each connected component, we determine whether it is a path or a cycle, compute their number of vertices, and sum their vertex cover numbers using the previous lemmas. Each of these steps takes polynomial time.

Exercise 3. A vertex cover C of a graph G is *minimal* if no strict subset of C is a vertex cover. Show that any graph has at most 2^k minimal vertex covers of size at most k. Furthermore, show that given G and k, all minimal vertex covers of G of size at most k can be enumerated in time $2^k n^{O(1)}$.

Solution sketch. write a procedure to check whether a vertex cover is minimal in polynomial time; adapt Algorithm `vc1` to enumerate all minimal vertex covers of size at most k.