1a. Introduction

COMP6741: Parameterized and Exact Computation

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19T3

Contents

1 Algorithms for NP-hard problems

Central question

P vs. NP

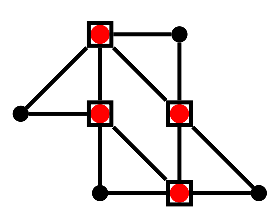
NP-hard problems

- no known polynomial time algorithm for any NP-hard problem
- belief: P $\neq$ NP
- What to do when facing an NP-hard problem?

Example problem: Vertex Cover
A *vertex cover* in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that every edge of G has an endpoint in S.

Vertex Cover	
Input:	Graph G, integer k
Question:	Does G have a vertex cover of size k?

Note: Vertex Cover is NP-complete.



Coping with NP-hardness

- Approximation algorithms
  - There is a polynomial-time algorithm, which, given a graph G, finds a vertex cover of G of size at most $2 \cdot \mathrm{OPT}$, where OPT is the size of a smallest vertex cover of G.
- Exact exponential time algorithms
  - There is an algorithm solving Vertex Cover in time $O(1.1970^n)$, where $n = |V|$.
- Fixed parameter algorithms
  - There is an algorithm solving Vertex Cover in time $O(1.2738^k + kn)$.
- Heuristics
  - The COVER heuristic (COVer Edges Randomly) finds a smaller vertex cover than state-of-the-art heuristics on a suite of hard benchmark instances.
- Restricting the inputs
  - Vertex Cover can be solved in polynomial time on bipartite graphs, trees, interval graphs, etc.
- Quantum algorithms?
  - Not believed to solve NP-hard problems in polynomial time.

Aims of this course

Design and analyze algorithms for NP-hard problems.

We focus on algorithms that solve NP-hard problems *exactly* and analyze their *worst case running time*.

2 Exponential Time Algorithms

Running times

Worst case running time of an algorithm.

- An algorithm is *polynomial* if $\exists c \in \mathbb{N}$ such that the algorithm solves every instance in time $O(n^c)$, where n is the size of the instance. Also: $n^{O(1)}$ or $\mathrm{poly}(n)$.
- *quasi-polynomial*: $2^{O(\log^c n)}$, $c \in O(1)$
- *sub-exponential*: $2^{o(n)}$
- *exponential*: $2^{\mathrm{poly}(n)}$
- *double-exponential*: $2^{2^{\mathrm{poly}(n)}}$

O^*-notation ignores polynomial factors in the input size:

$$O^*(f(n)) \equiv O(f(n) \cdot \mathsf{poly}(n))$$
$$O^*(f(k)) \equiv O(f(k) \cdot \mathsf{poly}(n))$$

Brute-force algorithms for NP-hard problems

Theorem 1. *Every problem in* NP *can be solved in exponential time.*

For a proof, see Lecture 1b on NP-completeness.

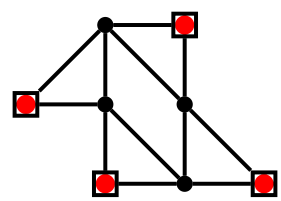
Three main categories for NP-complete problems

- Subset problems
- Permutation problems
- Partition problems

Subset Problem: Independent Set

An *independent set* in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that the vertices in S are pairwise non-adjacent in G.

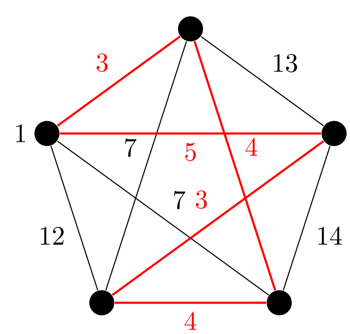
Independent Set	
Input:	Graph G, integer k
Question:	Does G have an independent set of size k?



Brute-force: $O^*(2^n)$, where $n = |V(G)|$

Permutation Problem: Traveling Salesman

Traveling Salesman Problem (TSP)	
Input:	a set of n cities, the distance $d(i, j) \in \mathbb{N}$ between every two cities i and j, integer k
Question:	Is there a permutation of the cities (a *tour*) such that the total distance when traveling from city to city in the specified order, and returning back to the origin, is at most k?

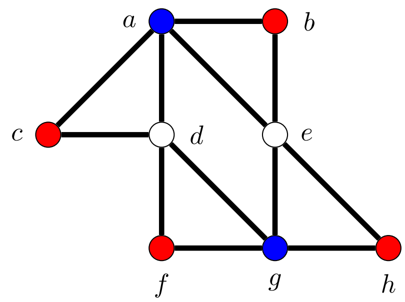


Brute-force: $O^*(n!) \subseteq 2^{O(n \log n)}$

Partition Problem: Coloring

A *k-coloring* of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, ..., k\}$ assigning colors to V such that no two adjacent vertices receive the same color.

Coloring	
Input:	Graph G, integer k
Question:	Does G have a k-coloring?



Brute-force: $O^*(k^n)$, where $n = |V(G)|$

Exponential Time Algorithms

- natural question in Algorithms: design faster (worst-case analysis) algorithms for problems
- might lead to practical algorithms
  - for small instances
    - you don't want to design software where your client/boss can find with better solutions *by hand* than your software
  - subroutines for
    - (sub)exponential time approximation algorithms
    - randomized algorithms with expected polynomial run time

Solve an NP-hard problem

- exhaustive search
  - trivial method
  - try all candidate solutions (certificates) for a ground set on n elements
  - running times for problems in NP
    - Subset Problems: $O^*(2^n)$
    - Permutation Problems: $O^*(n!)$
    - Partition Problems: $O^*(c^{n \log n})$
- faster exact algorithms
  - for some problems, it is possible to obtain provably faster algorithms
  - running times $O(1.0836^n), O(1.4689^n), O(1.9977^n)$

Exponential Time Algorithms in Practice

- How large are the instances one can solve in practice?

Available time nb. of operations	1 s 2^{36}	1 min 2^{42}	1 hour 2^{48}	3 days 2^{54}	6 months 2^{60}
n^5	147	337	776	1782	4096
n^{10}	12	18	27	42	64
1.05^n	511	596	681	767	852
1.1^n	261	305	349	392	436
1.5^n	61	71	82	92	102
2^n	36	42	48	54	60
5^n	15	18	20	23	25
$n!$	13	15	16	18	19

Note: Intel Core i7 920 (Quad core) executes between 2^{36} and 2^{37} instructions per second at 2.66 GHz.

> "For every polynomial-time algorithm you have, there is an exponential algorithm that I would rather run."
>
> – Alan Perlis (1922-1990, programming languages, 1st recipient of Turing Award)

Hardware vs. Algorithms

- Suppose a 2^n algorithm enables us to solve instances up to size x
- Faster processors
  - processor speed doubles after 18–24 months (Moore's law)
  - can solve instances up to size $x + 1$
- Faster algorithm
  - design an $O^*(2^{n/2}) \subseteq O(1.4143^n)$ time algorithm
  - can solve instances up to size $2 \cdot x$

3 Parameterized Complexity

A story

A computer scientist meets a biologist ... The biologist has performed n experiments. Unfortunately, the data obtained from these experiments has some conflicts. He suspects that a small number k of experiments have gone wrong, and he would like to detect whether removing k experiments can solve all the conflicts.

Eliminating conflicts from experiments

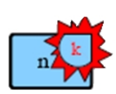
$n = 1000$ experiments, $k = 20$ experiments failed

Theoretical	Running Time Number of Instructions	Real
2^n	$1.07 \cdot 10^{301}$	$4.941 \cdot 10^{282}$ years
n^k	10^{60}	$4.611 \cdot 10^{41}$ years
$2^k \cdot n$	$1.05 \cdot 10^9$	0.01526 seconds

Notes:
– We assume that 2^{36} instructions are carried out per second.
– The Big Bang happened roughly $13.5 \cdot 10^9$ years ago.

Goal of Parameterized Complexity

Confine the combinatorial explosion to a parameter k.



For which problem–parameter combinations can we find algorithms with running times of the form

$$f(k) \cdot n^{O(1)},$$

where the f is a computable function independent of the input size n?

Examples of Parameters

A Parameterized Problem
Input: an instance of the problem
Parameter: a parameter k
Question: a Yes/No question about the instance and the parameter

- A parameter can be
  - input size (trivial parameterization)
  - solution size
  - related to the structure of the input (maximum degree, treewidth, branchwidth, genus, ...)
  - etc.

Main Complexity Classes

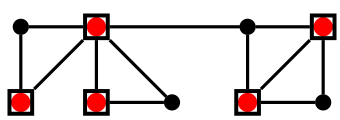
P: class of problems that can be solved in time $n^{O(1)}$
FPT: class of problems that can be solved in time $f(k) \cdot n^{O(1)}$
W[$\cdot$]: parameterized intractability classes
XP: class of problems that can be solved in time $f(k) \cdot n^{g(k)}$

$$\text{P} \subseteq \text{FPT} \subseteq \text{W}[1] \subseteq \text{W}[2] \cdots \subseteq \text{W}[P] \subseteq \text{XP}$$

Known: If FPT = W[1], then the Exponential Time Hypothesis fails, i.e. 3-Sat can be solved in time $2^{o(n)}$.

3.1 FPT Algorithm for Vertex Cover

Vertex Cover (VC)
Input: A graph $G = (V, E)$ on n vertices, an integer k
Parameter: k
Question: Is there a set of vertices $C \subseteq V$ of size at most k such that every edge has at least one endpoint in C?



3.2 Algorithms for Vertex Cover

Brute Force Algorithms

- $2^n \cdot n^{O(1)}$ not FPT
- $n^k \cdot n^{O(1)}$ not FPT

An FPT Algorithm

Algorithm vc1(G, k);

if $E = \emptyset$ **then** // all edges are covered
  return Yes
else if $k \leq 0$ **then** // we cannot select any vertex
  return No
else
  Select an edge $uv \in E$;
  return vc1$(G - u, k - 1)$ $\vee$ vc1$(G - v, k - 1)$

Running Time Analysis

- Let us look at an arbitrary execution of the algorithm.
- Recursive calls form a *search tree* T
  - with depth $\leq k$
  - where each node has ≤ 2 children
- $\Rightarrow T$ has $\leq 2^k$ leaves and $\leq 2^k - 1$ internal nodes
- at each node the algorithm spends time $n^{O(1)}$
- The running time is $O^*(2^k)$

A faster FPT Algorithm

Algorithm vc2(G, k);

```
if E = ∅ then                                    // all edges are covered
    return Yes
else if k ≤ 0 then                               // we used too many vertices
    return No
else if Δ(G) ≤ 2 then                            // G has maximum degree ≤ 2
    Solve the problem in polynomial time;
else
    Select a vertex v of maximum degree;
    return vc2(G − v, k − 1) ∨ vc2(G − N[v], k − d(v))
```

Running time analysis of vc2

- Number of leaves of the search tree:

$$\begin{aligned} T(k) &\leq T(k-1) + T(k-3) \\ x^k &\leq x^{k-1} + x^{k-3} \\ x^3 - x^2 - 1 &\leq 0 \end{aligned}$$

- The equation $x^3 - x^2 - 1 = 0$ has a unique positive real solution: $x \approx 1.4655...$
- Running time: $1.4656^k \cdot n^{O(1)}$

4 Administrivia

- Enrolments
- Website
- Lectures; exercises, questions, consultations
- Breaks
- Survey
- Course and assignment schedule
- Mid-session quiz
- Lecture recordings
- Glossary

5 Further Reading

- Exponential-time algorithms
  - Chapter 1, *Introduction*, in [FK10].
  - Woeginger's 2001 survey on exponential-time algorithms [Woe01].
  - Chapter 1, *Introduction*, in [Gas10].
- Parameterized Complexity
  - Chapter 1, *Introduction*, in [Cyg+15]
  - Chapter 2, *The Basic Definitions*, in [DF13]
  - Chapter I, *Foundations*, in [Nie06]
  - *Preface* in [FG06]

References

[Cyg+15] Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2015.

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[FG06] Jörg Flum and Martin Grohe. *Parameterized Complexity Theory*. Springer, 2006.

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[Gas10] Serge Gaspers. *Exponential Time Algorithms: Structures, Measures, and Bounds*. VDM Verlag Dr. Mueller, 2010.

[Nie06] Rolf Niedermeier. *Invitation to Fixed Parameter Algorithms*. Oxford University Press, 2006.

[Woe01] Gerhard J. Woeginger. "Exact Algorithms for NP-Hard Problems: A Survey". In: *Combinatorial Optimization* (2001), pp. 185–208.