8a. Randomized Algorithms

COMP6741: Parameterized and Exact Computation

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Outline

1. Introduction
2. Vertex Cover
3. Feedback Vertex Set
4. Color Coding
5. Monotone Local Search
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1. Introduction
2. Vertex Cover
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5. Monotone Local Search
Turing machines do not inherently have access to randomness.

Assume algorithm has also access to a stream of random bits drawn uniformly at random.

With $r$ random bits, the probability space is the set of all $2^r$ possible strings of random bits (with uniform distribution).
Las Vegas algorithms

Definition 1

A Las Vegas algorithm is a randomized algorithm whose output is always correct. Randomness is used to upper bound the expected running time of the algorithm.

Example

Quicksort with random choice of pivot.
**Definition 2**

- A **Monte Carlo algorithm** is an algorithm whose output is incorrect with probability at most $p$, $0 < p < 1$.
- A Monte Carlo has **one sided** error if its output is incorrect only on **Yes**-instances or on **No**-instances, but not both.
- A one-sided error Monte Carlo algorithm with **false negatives** answers **No** for every **No**-instance, and answers **Yes** on **Yes**-instances with probability $p \in (0, 1)$. We say that $p$ is the **success probability** of the algorithm.
Suppose \( A \) is a one-sided Monte Carlo algorithm with false negatives with success probability \( p \). How can we use \( A \) to design a new one-sided Monte Carlo algorithm with success probability \( p^* > p \)?

Let \( t = -\ln(1 - p^*) \) and run the algorithm \( t \) times. Return Yes if at least one run of the algorithm returned Yes, and No otherwise.

Failure probability is \((1 - p^*)^t \leq e^{-pt} = e^{-\ln(1 - p^*)} = 1 - p^*\) via the inequality \(1 - x \leq e^{-x}\).
Algorithms with increased success probability

Boosting success probability

Suppose $A$ is a one-sided Monte Carlo algorithm with false negatives with success probability $p$. How can we use $A$ to design a new one-sided Monte Carlo algorithm with success probability $p^* > p$?

Let $t = -\frac{\ln(1-p^*)}{p}$ and run the algorithm $t$ times. Return **Yes** if at least one run of the algorithm returned **Yes**, and **No** otherwise.
Boosting success probability

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Let $t = -\frac{\ln(1-p^*)}{p}$ and run the algorithm $t$ times. Return $\text{YES}$ if at least one run of the algorithm returned $\text{YES}$, and $\text{NO}$ otherwise. Failure probability is

$$(1 - p)^t \leq (e^{-p})^t = \frac{1}{e^{pt}} = e^{\ln(1-p^*)} = 1 - p^*$$

via the inequality $1 - x \leq e^{-x}$. 

Algorithms with increased success probability

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Suppose $A$ is a one-sided Monte Carlo algorithm with false negatives with success probability $p$. How can we use $A$ to design a new one-sided Monte Carlo algorithm with success probability $p^* > p$?

Let $t = -\frac{\ln(1-p^*)}{p}$ and run the algorithm $t$ times. Return $\text{YES}$ if at least one run of the algorithm returned $\text{YES}$, and $\text{NO}$ otherwise. Failure probability is

$$ (1 - p)^t \leq (e^{-p})^t = \frac{1}{e^{pt}} = e^{\ln(1-p^*)} = 1 - p^* $$

via the inequality $1 - x \leq e^{-x}$.

Definition 3

A randomized algorithm is a one-sided Monte Carlo algorithm with constant success probability.
Theorem 4

If a one-sided error Monte Carlo algorithm has success probability at least $p$, then repeating it independently $\lceil \frac{1}{p} \rceil$ times gives constant success probability.

In particular if we have a polynomial-time one-sided error Monte Carlo algorithm with success probability $p = \frac{1}{f(k)}$ for some computable function $f$, then we get a randomized FPT algorithm with running time $O^*(f(k))$. 
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Vertex Cover

For a graph $G = (V, E)$ a vertex cover $X \subseteq V$ is a set of vertices such that every edge is adjacent to a vertex in $X$.

**Vertex Cover**

Input: Graph $G$, integer $k$

Parameter: $k$

Question: Does $G$ have a vertex cover of size $k$?
For a graph $G = (V, E)$ a vertex cover $X \subseteq V$ is a set of vertices such that every edge is adjacent to a vertex in $X$.

**Vertex Cover**

**Input:** Graph $G$, integer $k$

**Parameter:** $k$

**Question:** Does $G$ have a vertex cover of size $k$?

**Warm-up:** design a randomized algorithm with running time $O^*(2^k)$. 
Theorem 5

**Vertex Cover** has a randomized algorithm with running time $O^*(2^k)$.

Proof.

- Select an edge $uv \in E$ uniformly at random.
- Select an endpoint $w \in \{u, v\}$ of that edge uniformly at random.
- Add $w$ to the partial vertex cover $S$ (initially empty).
- If $G$ has vertex cover number at most $k$, then repeating this $k$ times gives a vertex cover with probability at least $\frac{1}{2^k}$.
- Applying Theorem 4 gives a randomized FPT running time of $O^*(2^k)$. 

□
Outline

1 Introduction

2 Vertex Cover

3 Feedback Vertex Set

4 Color Coding

5 Monotone Local Search
A \textit{feedback vertex set} of a multigraph $G = (V, E)$ is a set of vertices $S \subset V$ such that $G - S$ is acyclic.

<table>
<thead>
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<th>Feedback Vertex Set</th>
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A *feedback vertex set* of a multigraph $G = (V, E)$ is a set of vertices $S \subset V$ such that $G - S$ is acyclic.

**Feedback Vertex Set**

**Input:** Multigraph $G$, integer $k$

**Parameter:** $k$

**Question:** Does $G$ have a feedback vertex of size $k$?

Recall our simplification rules for Feedback Vertex Set.
Simplification Rules

1. Loop: If loop at vertex $v$, remove $v$ and decrease $k$ by 1
2. Multiedge: Reduce the multiplicity of each edge with multiplicity $\geq 3$ to 2.
3. Degree-1: If $v$ has degree at most 1 then remove $v$.
4. Degree-2: If $v$ has degree 2 with neighbors $u, w$ then delete 2 edges $uv, vw$ and replace with new edge $uw$.
5. Budget: If $k < 0$, then return no.
The solution is incident to a constant fraction of the edges

Lemma 6

Let $G$ be a multigraph with minimum degree at least 3. Then, for every feedback vertex set $X$ of $G$, at least $\frac{1}{3}$ of the edges have at least one endpoint in $X$. 
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Let $G$ be a multigraph with minimum degree at least 3. Then, for every feedback vertex set $X$ of $G$, at least $1/3$ of the edges have at least one endpoint in $X$.

Proof.

Denote by $n$ and $m$ the number of vertices and edges of $G$, respectively. Since $\delta(G) \geq 3$, we have that $m \geq 3n/2$. Let $F := G - X$. Since $F$ has at most $n - 1$ edges, at least $\frac{1}{3}$ of the edges have an endpoint in $X$. 

\qed
Theorem 7

**Feedback Vertex Set** has a randomized algorithm with running time $O^*(6^k)$. 

We prove the theorem using the following algorithm.

$S \leftarrow \emptyset$

Do $k$ times: Apply simplification rules; add a random endpoint of a random edge to $S$.

If $S$ is a feedback vertex set, return Yes, otherwise return No.
Feedback Vertex Set has a randomized algorithm with running time $O^*(6^k)$.

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- $S \leftarrow \emptyset$
- Do $k$ times: Apply simplification rules; add a random endpoint of a random edge to $S$.
- If $S$ is a feedback vertex set, return Yes, otherwise return No.
Proof.

We need to show: each time the algorithm adds a vertex $v$ to $S$, if $(G - S, k - |S|)$ is a Yes-instance, then with probability at least $1/6$, the instance $(G - (S \cup \{v\}), k - |S| - 1)$ is also a Yes-instance. Then, by induction, we can conclude that with probability $1/(6^k)$, the algorithm finds a feedback vertex set of size at most $k$ if it is given a Yes-instance.
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- We need to show: each time the algorithm adds a vertex \( v \) to \( S \), if \((G - S, k - |S|)\) is a \textbf{Yes}-instance, then with probability at least \(1/6\), the instance \((G - (S \cup \{v\}), k - |S| - 1)\) is also a \textbf{Yes}-instance. Then, by induction, we can conclude that with probability \(1/(6^k)\), the algorithm finds a feedback vertex set of size at most \(k\) if it is given a \textbf{Yes}-instance.

- Assume \((G - S, k - |S|)\) is a \textbf{Yes}-instance.

- Lemma 6 implies that with probability at least \(1/3\), a randomly chosen edge \(uv\) has at least one endpoint in some feedback vertex set of size \(k - |S|\).

- So, with probability at least \(\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}\), a randomly chosen endpoint of \(uv\) belongs some feedback vertex set.
We need to show: each time the algorithm adds a vertex \( v \) to \( S \), if \((G - S, k - |S|)\) is a \textbf{Yes}-instance, then with probability at least \( 1/6 \), the instance \((G - (S \cup \{v\}), k - |S| - 1)\) is also a \textbf{Yes}-instance. Then, by induction, we can conclude that with probability \( 1/(6^k) \), the algorithm finds a feedback vertex set of size at most \( k \) if it is given a \textbf{Yes}-instance.

Assume \((G - S, k - |S|)\) is a \textbf{Yes}-instance.

Lemma 6 implies that with probability at least \( 1/3 \), a randomly chosen edge \( uv \) has at least one endpoint in some feedback vertex set of size \( k - |S| \).

So, with probability at least \( \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6} \), a randomly chosen endpoint of \( uv \) belongs some feedback vertex set.

Applying Theorem 4 gives a randomized FPT running time of \( O^*(6^k) \).
Improved analysis

Lemma 8

Let $G$ be a multigraph with minimum degree at least 3. For every feedback vertex set $X$, at least $\frac{1}{2}$ of the edges of $G$ have at least one endpoint in $X$. 

Note: For a feedback vertex set $X$, consider the forest $F := G - X$. The statement is equivalent to:

$|E(G) \setminus E(F)| \geq |E(F)|$

Let $J \subseteq E(G)$ denote the edges with one endpoint in $X$, and the other in $V(F)$.

We will show the stronger result:

$|J| \geq |V(F)|$
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Improved analysis

Proof.

Let $V_{\leq 1}, V_2, V_{\geq 3}$ be the set of vertices that have degree at most 1, exactly 2, and at least 3, respectively, in $F$. 

Since $\delta(G) \geq 3$, each vertex in $V_{\leq 1}$ contributes at least 2 edges to $J$, and each vertex in $V_2$ contributes at least 1 edge to $J$. We show that $|V_{\geq 3}| \leq |V_{\leq 1}|$ by induction on $|V(F)|$. Trivially true for forests with at most 1 vertex. Assume true for forests with at most $n-1$ vertices. For any forest on $n$ vertices, consider removing a leaf (which must always exist) to obtain $F'$ with the vertex partition $(V_{\leq 1}', V_2', V_{\geq 3}')$. If $|V_{\geq 3}| = |V_{\geq 3}'|$, then we have that $|V_{\geq 3}| = |V_{\geq 3}'| \leq |V_{\leq 1}'| \leq |V_{\geq 1}|$. Otherwise, $|V_{\geq 3}| = |V_{\geq 3}'| + 1 \leq |V_{\leq 1}'| + 1 = |V_{\leq 1}|$. We conclude that:

$|E(G) \setminus E(F)| \geq |J| \geq 2 |V_{\leq 1}| + |V_2| + |V_{\geq 3}| = |V(F)|$.
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  - For any forest on $n$ vertices, consider removing a leaf (which must always exist) to obtain $F'$ with the vertex partition $(V'_{\leq 1}, V'_2, V'_{\geq 3})$.
    - If $|V_{\geq 3}| = |V'_{\geq 3}|$, then we have that $|V_{\geq 3}| = |V'_{\geq 3}| \leq |V'_{\leq 1}| \leq |V_{\leq 1}|$.
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**Proof.**

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  - For any forest on $n$ vertices, consider removing a leaf (which must always exist) to obtain $F'$ with the vertex partition $(V'_{\leq 1}, V'_2, V'_{\geq 3})$.
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- We conclude that:

$$|E(G) \setminus E(F)| \geq |J| \geq 2|V_{\leq 1}| + |V_2| \geq |V_{\leq 1}| + |V_2| + |V_{\geq 3}| = |V(F)|$$
Improved Randomized Algorithm

Theorem 9

**Feedback Vertex Set** has a randomized algorithm with running time $O^*(4^k)$.

Note

This algorithmic method is applicable whenever the vertex set we seek is incident to a constant fraction of the edges.
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**Longest Path**

**Input:** Graph $G$, integer $k$

**Parameter:** $k$

**Question:** Does $G$ have a path on $k$ vertices as a subgraph?

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**NP-complete**

To show that *Longest Path* is NP-hard, reduce from *Hamiltonian Path* by setting $k = n$ and leaving the graph unchanged.
**Longest Path**

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**NP-complete**

To show that Longest Path is NP-hard, reduce from Hamiltonian Path by setting $k = n$ and leaving the graph unchanged.
Lemma 10

Let $U$ be a set of size $n$, and let $X \subseteq U$ be a subset of size $k$. Let $\chi : U \rightarrow [k]$ be a coloring of the elements of $U$, chosen uniformly at random. Then the probability that the elements of $X$ are colored with pairwise distinct colors is at least $e^{-k}$.
Lemma 10

Let $U$ be a set of size $n$, and let $X \subseteq U$ be a subset of size $k$. Let $\chi : U \rightarrow [k]$ be a coloring of the elements of $U$, chosen uniformly at random. Then the probability that the elements of $X$ are colored with pairwise distinct colors is at least $e^{-k}$.

Proof.

There are $k^n$ possible colorings $\chi$ and $k!k^{n-k}$ of them are injective on $X$. Using the inequality

$$k! > (k/e)^k,$$

the lemma follows since

$$\frac{k! \cdot k^{n-k}}{k^n} > \frac{k^k \cdot k^{n-k}}{e^k \cdot k^n} = e^{-k}.$$
A path is **colorful** if all vertices of the path are colored with pairwise distinct colors.

**Lemma 11**

Let $G$ be an undirected graph, and let $\chi : V(G) \rightarrow [k]$ be a coloring of its vertices with $k$ colors. There is an algorithm that checks in time $O^*(2^k)$ whether $G$ contains a colorful path on $k$ vertices.
Proof.

Partition $V(G)$ into $V_1, \ldots, V_k$ subsets such that vertices in $V_i$ are colored $i$. 
Proof.

Partition $V(G)$ into $V_1, \ldots, V_k$ subsets such that vertices in $V_i$ are colored $i$. Apply dynamic programming on nonempty $S \subseteq \{1, \ldots, k\}$. For $u \in \bigcup_{i \in S} V_i$ let $P(S, u) = true$ if there is a colorful path with colors from $S$ and $u$ as an endpoint.
Colorful Path II

Proof.

Partition $V(G)$ into $V_1, ..., V_k$ subsets such that vertices in $V_i$ are colored $i$. Apply dynamic programming on nonempty $S \subseteq \{1, ..., k\}$. For $u \in \bigcup_{i \in S} V_i$ let $P(S, u) = true$ if there is a colorful path with colors from $S$ and $u$ as an endpoint. We have the following:

- For $|S| = 1$, $P(S, u) = true$ for $u \in V(G)$ iff $S = \{\chi(u)\}$.
- For $|S| > 1$

$$P(S, u) = \begin{cases} \bigvee_{uv \in E(G)} P(S \setminus \{\chi(u)\}, v) & \text{if } \chi(u) \in S \\ false & \text{otherwise} \end{cases}$$
Proof.

Partition $V(G)$ into $V_1, ..., V_k$ subsets such that vertices in $V_i$ are colored $i$. Apply dynamic programming on nonempty $S \subseteq \{1, ..., k\}$. For $u \in \bigcup_{i \in S} V_i$ let $P(S, u) = true$ if there is a colorful path with colors from $S$ and $u$ as an endpoint. We have the following:

- For $|S| = 1$, $P(S, u) = true$ for $u \in V(G)$ iff $S = \{\chi(u)\}$.
- For $|S| > 1$

$$P(S, u) = \begin{cases} \bigvee_{uv \in E(G)} P(S \setminus \{\chi(u)\}, v) & \text{if } \chi(u) \in S \\ false & \text{otherwise} \end{cases}$$

All values of $P$ can be computed in $O^*(2^k)$ time and there exists a colorful $k$-path iff $P([k], v)$ is true for some vertex $v \in V(G)$. \qed
Theorem 12

\textbf{Longest Path} has a randomized algorithm with running time $O^*((2 \cdot e)^k)$.

Note

This algorithmic method is applicable whenever we seek a vertex set of size $O(f(k))$ that has constant treewidth.
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Exponential-time algorithms and parameterized algorithms

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**Exponential-time algorithms**

- Algorithms for NP-hard problems
- Beat brute-force & improve
- Running time measured in the size of the universe $n$
- $O(2^n \cdot n), O(1.5086^n), O(1.0892^n)$

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**Parameterized algorithms**

- Algorithms for NP-hard problems
- Use a parameter $k$ (often $k$ is the solution size)
- Algorithms with running time $f(k) \cdot n^c$
- $k^k n^{O(1)}, 5^k n^{O(1)}, O(1.2738^k + kn)$

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Can we use Parameterized algorithms to design fast Exponential-time algorithms?
Example: Feedback Vertex Set

$S \subseteq V$ is a feedback vertex set in a graph $G = (V, E)$ if $G - S$ is acyclic.

**Feedback Vertex Set**

Input: Graph $G = (V, E)$, integer $k$

Parameter: $k$

Question: Does $G$ have a feedback vertex set of size at most $k$?
Example: Feedback Vertex Set

\[ S \subseteq V \text{ is a feedback vertex set in a graph } G = (V, E) \text{ if } G - S \text{ is acyclic.} \]

**Feedback Vertex Set**

**Input:** Graph \( G = (V, E) \), integer \( k \)

**Parameter:** \( k \)

**Question:** Does \( G \) have a feedback vertex set of size at most \( k \)?

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**Exponential-time algorithms**

- \( O^*(2^n) \) trivial
- \( O(1.8899^n) \) [Razgon 06]
- \( O(1.7548^n) \) [Fomin, Gaspers, Pyatkin 06]
- \( O(1.7356^n) \) [Xiao, Nagamoshi 13]
- \( O(1.7347^n) \) [Fomin, Todinca, Villanger 15]

**Parameterized algorithms**

- \( O^*((17k^4)!|Bodlaender 94|) \)
- \( O^*((2k + 1)^k) \) [Downey, Fellows 98]
- \( O^*(3.591^k) \) [Kociumaka, Pilipczuk 14]
- \( O^*(3^k) \) (r) [Cygan 11]
Exponential-time algorithms via parameterized algorithms

Binomial coefficients

\[ \arg \max_{0 \leq k \leq n} \binom{n}{k} = \frac{n}{2} \quad \text{and} \quad \binom{n}{n/2} = \Theta(2^n / \sqrt{n}) \]
Binomial coefficients

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\arg \max_{0 \leq k \leq n} \binom{n}{k} = \frac{n}{2} \quad \text{and} \quad \binom{n}{n/2} = \Theta(2^{n/\sqrt{n}})
\]

Algorithm for Feedback Vertex Set

- Set \( t = 0.60909 \cdot n \)
- If \( k \leq t \), run \( O^*(3^k) \) algorithm
- Else check all \( \binom{n}{k} \) vertex subsets of size \( k \)

Running time: \( O^* \left( \max \left( 3^t, \binom{n}{t} \right) \right) = O^*(1.9526^n) \)
Exponential-time algorithms via parameterized algorithms

**Binomial coefficients**

$$\arg \max_{0 \leq k \leq n} \binom{n}{k} = n/2 \quad \text{and} \quad \binom{n}{n/2} = \Theta(2^{n/\sqrt{n}})$$

**Algorithm for Feedback Vertex Set**

- Set $t = 0.60909 \cdot n$
- If $k \leq t$, run $O^*(3^k)$ algorithm
- Else check all $\binom{n}{k}$ vertex subsets of size $k$

Running time: $O^*(\max(3^t, \binom{n}{t})) = O^*(1.9526^n)$

This approach gives algorithms faster than $O^*(2^n)$ for subset problems with a parameterized algorithm faster than $O^*(4^k)$. 
An *implicit set system* is a function $\Phi$ with:

- **Input**: instance $I \in \{0, 1\}^*$, $|I| = N$
- **Output**: set system $(U_I, F_I)$:
  - universe $U_I$, $|U_I| = n$
  - family $F_I$ of subsets of $U_I$
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**$\Phi$-SUBSET**

**Input:** Instance $I$

**Question:** Is $|F_I| > 0$?
An *implicit set system* is a function $\Phi$ with:

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### $\Phi$-Subset

**Input:** Instance $I$

**Question:** Is $|F_I| > 0$?

### $\Phi$-Extension

**Input:** Instance $I$, a set $X \subseteq U_I$, and an integer $k$

**Question:** Does there exist a subset $S \subseteq (U_I \setminus X)$ such that $S \cup X \in F_I$ and $|S| \leq k$?
Suppose \( \Phi \text{-EXTENSION} \) has a \( O^*(c^k) \) time algorithm \( B \).

**Algorithm for checking whether \( \mathcal{F}_I \) contains a set of size \( k \)**

- Set \( t = \max \left( 0, \frac{ck-n}{c-1} \right) \)
- Uniformly at random select a subset \( X \subseteq U_I \) of size \( t \)
- Run \( B(I, X, k-t) \)
Suppose $\Phi$-extension has a $O^*(c^k)$ time algorithm $B$.

### Algorithm for checking whether $\mathcal{F}_I$ contains a set of size $k$

- Set $t = \max \left(0, \frac{ck-n}{c-1}\right)$
- Uniformly at random select a subset $X \subseteq U_I$ of size $t$
- Run $B(I, X, k-t)$

**Running time:** [Fomin, Gaspers, Lokshtanov, Saurabh 16]

\[
O^* \left( \frac{n}{t} \cdot \frac{c^{k-t}}{k^t} \right) = O^* \left( 2 - \frac{1}{c} \right)^n
\]
Intuition

Brute-force randomized algorithm

- Pick \( k \) elements of the universe one-by-one.
- Suppose \( \mathcal{F}_I \) contains a set of size \( k \).

Success probability:

\[
\frac{k}{n} \cdot \frac{k-1}{n-1} \cdot \frac{k-t}{n-t} \cdot \ldots \cdot \frac{2}{n-(k-2)} \cdot \frac{1}{n-(k-1)} = \frac{1}{\binom{n}{k}}
\]

\[
\Rightarrow \quad \frac{1}{c}
\]
Randomized Monotone Local Search

Theorem 13 ([Fomin, Gaspers, Lokshtanov, Saurabh 16])

If there exists a (randomized) algorithm for $\Phi$-Extension with running time $O^*(c^k)$ then there exists a randomized algorithm for $\Phi$-Subset with running time $(2 - \frac{1}{c})^n \cdot N^{O(1)}$. 
Randomized Monotone Local Search

Theorem 13 ([Fomin, Gaspers, Lokshtanov, Saurabh 16])

If there exists a (randomized) algorithm for $\Phi$-Extension with running time $O^*(c^k)$ then there exists a randomized algorithm for $\Phi$-Subset with running time $(2 - \frac{1}{c})^n \cdot N^{O(1)}$.

Theorem 14 ([Fomin, Gaspers, Lokshtanov, Saurabh 16])

Feedback Vertex Set has a randomized algorithm with running time $O^* \left( (2 - \frac{1}{3})^n \right) \subseteq O(1.6667^n)$.
Derandomization at the expense of a subexponential factor in the running time.

**Theorem 15 ([Fomin, Gaspers, Lokshtanov, Saurabh 16])**

*If there exists an algorithm for $\Phi$-Extension with running time $O^*(c^k)$ then there exists an algorithm for $\Phi$-Subset with running time $(2 - \frac{1}{c})^{n+o(n)} \cdot N^{O(1)}$.***
Derandomization at the expense of a subexponential factor in the running time.

**Theorem 15 ([Fomin, Gaspers, Lokshtanov, Saurabh 16])**

If there exists an algorithm for \( \Phi\text{-EXTENSION} \) with running time \( O^*(c^k) \) then there exists an algorithm for \( \Phi\text{-SUBSET} \) with running time \( (2 - \frac{1}{c})^{n+o(n)} \cdot N^O(1) \).

**Theorem 16 ([Fomin, Gaspers, Lokshtanov, Saurabh 16])**

**Feedback Vertex Set** has an algorithm with running time \( O^* \left( \left(2 - \frac{1}{3.591} \right)^n \right) \subseteq O(1.7216^n) \).
Multivariate Subroutines

**Theorem 17 ([Gaspers, Lee 17])**

*If there exists an algorithm for Φ-Extension with running time* $O^*(b^n - |X| \cdot c^k)$ *
*then there exists an algorithm for Φ-Subset with running time*

$(1 + b - \frac{1}{c})^{n+o(n)} \cdot N^{O(1)}$.

**Theorem 18 ([Gaspers, Lee 17])**

*Feedback Vertex Set Extension can be solved in time* $O(1.5422^n - |X| 1.2041^k)$.

**Corollary 19 ([Gaspers, Lee 17])**

*Feedback Vertex Set can be solved in time* $O(1.7117^n)$.
Further Reading

