8a. Randomized Algorithms
COMP6741: Parameterized and Exact Computation

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Outline

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Randomized Algorithms

- Turing machines do not inherently have access to randomness.
- Assume algorithm has also access to a stream of random bits drawn uniformly at random.
- With r random bits, the probability space is the set of all 2^r possible strings of random bits (with uniform distribution).

Las Vegas algorithms

Definition 1

A Las Vegas algorithm is a randomized algorithm whose output is always correct.

Randomness is used to upper bound the expected running time of the algorithm.

Example

Quicksort with random choice of pivot.

Monte Carlo algorithms

Definition 2

- A Monte Carlo algorithm is an algorithm whose output is incorrect with probability at most p, $0 < p < 1$.
- A Monte Carlo has one sided error if its output is incorrect only on $\textsc{Yes}$-instances or on $\textsc{No}$-instances, but not both.
- A one-sided error Monte Carlo algorithm with false negatives answers $\textsc{No}$ for every $\textsc{No}$-instance, and answers $\textsc{Yes}$ on $\textsc{Yes}$-instances with probability $p \in (0, 1)$. We say that p is the *success probability* of the algorithm.

Algorithms with increased success probability

Boosting success probability

Suppose A is a one-sided Monte Carlo algorithm with false negatives with success probability p. How can we use A to design a new one-sided Monte Carlo algorithm with success probability $p^* > p$?

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$$(1-p)^t \leq (e^{-p})^t = \frac{1}{e^{pt}} = e^{\ln(1-p^*)} = 1 - p^*$$

via the inequality $1 - x \leq e^{-x}$.

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via the inequality $1 - x \leq e^{-x}$.

Definition 3

A randomized algorithm is a one-sided Monte Carlo algorithm with constant success probability.

Amplification

Theorem 4

If a one-sided error Monte Carlo algorithm has success probability at least p, then repeating it independently $\lceil \frac{1}{p} \rceil$ times gives constant success probability.

In particular if we have a polynomial-time one-sided error Monte Carlo algorithm with success probability $p = \frac{1}{f(k)}$ for some computable function f, then we get a randomized FPT algorithm with running time $O^*(f(k))$.

Outline

Vertex Cover

For a graph $G = (V, E)$ a vertex cover $X \subseteq V$ is a set of vertices such that every edge is adjacent to a vertex in X.

VERTEX COVER	
Input:	Graph G, integer k
Parameter:	k
Question:	Does G have a vertex cover of size k?

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Warm-up: design a randomized algorithm with running time $O^*(2^k)$.

Randomized Algorithm for Vertex Cover

Theorem 5

VERTEX COVER *has a randomized algorithm with running time* $O^*(2^k)$.

Proof.

- Select an edge $uv \in E$ uniformly at random.
- Select an endpoint $w \in \{u, v\}$ of that edge uniformly at random.
- Add w to the partial vertex cover S (initially empty).
- If G has vertex cover number at most k, then repeating this k times gives a vertex cover with probability at least $\frac{1}{2^k}$.
- Applying Theorem 4 gives a randomized FPT running time of $O^*(2^k)$.

□

Outline

Feedback Vertex Set

A *feedback vertex set* of a multigraph $G = (V, E)$ is a set of vertices $S \subset V$ such that $G - S$ is acyclic.

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Recall our simplification rules for FEEDBACK VERTEX SET.

Simplification Rules

1. Loop: If loop at vertex v, remove v and decrease k by 1
2. Multiedge: Reduce the multiplicity of each edge with multiplicity ≥ 3 to 2.
3. Degree-1: If v has degree at most 1 then remove v.
4. Degree-2: If v has degree 2 with neighbors u, w then delete 2 edges uv, vw and replace with new edge uw.
5. Budget: If $k < 0$, then return no.

The solution is incident to a constant fraction of the edges

Lemma 6

Let G be a multigraph with minimum degree at least 3. Then, for every feedback vertex set X of G, at least $1/3$ of the edges have at least one endpoint in X.

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Lemma 6

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Proof.

Denote by n and m the number of vertices and edges of G, respectively.
Since $\delta(G) \geq 3$, we have that $m \geq 3n/2$.
Let $F := G - X$.
Since F has at least most $n-1$ edges, at least $\frac{1}{3}$ of the edges have an endpoint in X. □

Randomized Algorithm

Theorem 7

FEEDBACK VERTEX SET *has a randomized algorithm with running time* $O^*(6^k)$.

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We prove the theorem using the following algorithm.

- $S \leftarrow \emptyset$
- Do k times: Apply simplification rules; add a random endpoint of a random edge to S.
- If S is a feedback vertex set, return YES, otherwise return NO.

Proof

Proof.

- We need to show: each time the algorithm adds a vertex v to S, if $(G - S, k - |S|)$ is a YES-instance, then with probability at least $1/6$, the instance $(G - (S \cup \{v\}), k - |S| - 1)$ is also a YES-instance. Then, by induction, we can conclude that with probability $1/(6^k)$, the algorithm finds a feedback vertex set of size at most k if it is given a YES-instance.

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- Assume $(G - S, k - |S|)$ is a Yes-instance.
- Lemma 6 implies that with probability at least $1/3$, a randomly chosen edge uv has at least one endpoint in some feedback vertex set of size $k - |S|$.
- So, with probability at least $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$, a randomly chosen endpoint of uv belongs some feedback vertex set.

Proof

Proof.

- We need to show: each time the algorithm adds a vertex v to S, if $(G - S, k - |S|)$ is a $\textsc{Yes}$-instance, then with probability at least $1/6$, the instance $(G - (S \cup \{v\}), k - |S| - 1)$ is also a $\textsc{Yes}$-instance. Then, by induction, we can conclude that with probability $1/(6^k)$, the algorithm finds a feedback vertex set of size at most k if it is given a $\textsc{Yes}$-instance.
- Assume $(G - S, k - |S|)$ is a $\textsc{Yes}$-instance.
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- So, with probability at least $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$, a randomly chosen endpoint of uv belongs some feedback vertex set.
- Applying Theorem 4 gives a randomized FPT running time of $O^*(6^k)$.

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Improved analysis

Lemma 8

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Note: For a feedback vertex set X, consider the forest $F := G - X$. The statement is equivalent to:

$$|E(G) \setminus E(F)| \geq |E(F)|$$

Let $J \subseteq E(G)$ denote the edges with one endpoint in X, and the other in $V(F)$. We will show the stronger result:

$$|J| \geq |V(F)|$$

Improved analysis

Proof.

- Let $V_{\leq 1}, V_2, V_{\geq 3}$ be the set of vertices that have degree at most 1, exactly 2, and at least 3, respectively, in F.

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- Since $\delta(G) \geq 3$, each vertex in $V_{\leq 1}$ contributes at least 2 edges to J, and each vertex in V_2 contributes at least 1 edge to J.

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- Let $V_{\leq 1}, V_2, V_{\geq 3}$ be the set of vertices that have degree at most 1, exactly 2, and at least 3, respectively, in F.
- Since $\delta(G) \geq 3$, each vertex in $V_{\leq 1}$ contributes at least 2 edges to J, and each vertex in V_2 contributes at least 1 edge to J.
- We show that $|V_{\geq 3}| \leq |V_{\leq 1}|$ by induction on $|V(F)|$.
  - Trivially true for forests with at most 1 vertex.
  - Assume true for forests with at most $n-1$ vertices.
  - For any forest on n vertices, consider removing a leaf (which must always exist) to obtain F' with the vertex partition $(V'_{\leq 1}, V'_2, V'_{\geq 3})$.
    If $|V_{\geq 3}| = |V'_{\geq 3}|$, then we have that $|V_{\geq 3}| = |V'_{\geq 3}| \leq |V'_{\leq 1}| \leq |V_{\geq 1}|$.
    Otherwise, $|V_{\geq 3}| = |V'_{\geq 3}| + 1 \leq |V'_{\leq 1}| + 1 = |V_{\leq 1}|$.

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    Otherwise, $|V_{\geq 3}| = |V'_{\geq 3}| + 1 \leq |V'_{\leq 1}| + 1 = |V_{\leq 1}|$.
- We conclude that:

$$|E(G) \setminus E(F)| \geq |J| \geq 2|V_{\leq 1}| + |V_2| \geq |V_{\leq 1}| + |V_2| + |V_{\geq 3}| = |V(F)|$$

$\square$

Improved Randomized Algorithm

Theorem 9

FEEDBACK VERTEX SET *has a randomized algorithm with running time* $O^*(4^k)$.

Note

This algorithmic method is applicable whenever the vertex set we seek is incident to a constant fraction of the edges.

Outline

Longest Path

LONGEST PATH

Input:	Graph G, integer k
Parameter:	k
Question:	Does G have a path on k vertices as a subgraph?

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NP-complete

To show that LONGEST PATH is NP-hard, reduce from HAMILTONIAN PATH by setting $k = n$ and leaving the graph unchanged.

Color Coding

Lemma 10

Let U be a set of size n, and let $X \subseteq U$ be a subset of size k. Let $\chi : U \to [k]$ be a coloring of the elements of U, chosen uniformly at random. Then the probability that the elements of X are colored with pairwise distinct colors is at least e^{-k}.

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Proof.

There are k^n possible colorings χ and $k!k^{n-k}$ of them are injective on X. Using the inequality

$$k! > (k/e)^k,$$

the lemma follows since

$$\frac{k! \cdot k^{n-k}}{k^n} > \frac{k^k \cdot k^{n-k}}{e^k \cdot k^n} = e^{-k}.$$

□

Colorful Path

A path is colorful if all vertices of the path are colored with pairwise distinct colors.

Lemma 11

Let G be an undirected graph, and let $\chi : V(G) \to [k]$ be a coloring of its vertices with k colors. There is an algorithm that checks in time $O^*(2^k)$ whether G contains a colorful path on k vertices.

Colorful Path II

Proof.

Partition $V(G)$ into $V_1, ..., V_k$ subsets such that vertices in V_i are colored i.

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Partition $V(G)$ into $V_1, ..., V_k$ subsets such that vertices in V_i are colored i. Apply dynamic programming on nonempty $S \subseteq \{1, ..., k\}$. For $u \in \bigcup_{i \in S} V_i$ let $P(S, u) = true$ if there is a colorful path with colors from S and u as an endpoint.

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- For $|S| = 1$, $P(S, u) = true$ for $u \in V(G)$ iff $S = \{\chi(u)\}$.
- For $|S| > 1$

$$P(S, u) = \begin{cases} \bigvee_{uv \in E(G)} P(S \setminus \{\chi(u)\}, v) & \text{if } \chi(u) \in S \\ false & \text{otherwise} \end{cases}$$

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$$P(S,u) = \begin{cases} \bigvee_{uv \in E(G)} P(S \setminus \{\chi(u)\}, v) & \text{if } \chi(u) \in S \\ false & \text{otherwise} \end{cases}$$

All values of P can be computed in $O^*(2^k)$ time and there exists a colorful k-path iff $P([k], v)$ is true for some vertex $v \in V(G)$. □

Longest Path

Theorem 12

LONGEST PATH *has a randomized algorithm with running time* $O^*((2 \cdot e)^k)$.

Note

This algorithmic method is applicable whenever we seek a vertex set of size $O(f(k))$ that has constant treewidth.

Outline

Exponential-time algorithms and parameterized algorithms

Exponential-time algorithms

- Algorithms for NP-hard problems
- Beat brute-force & improve
- Running time measured in the size of the universe n
- $O(2^n \cdot n)$, $O(1.5086^n)$, $O(1.0892^n)$

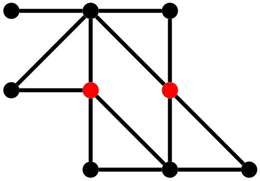
Parameterized algorithms

- Algorithms for NP-hard problems
- Use a parameter k (often k is the solution size)
- Algorithms with running time $f(k) \cdot n^c$
- $k^k n^{O(1)}$, $5^k n^{O(1)}$, $O(1.2738^k + kn)$

Can we use Parameterized algorithms to design fast Exponential-time algorithms?

Example: Feedback Vertex Set

$S \subseteq V$ is a feedback vertex set in a graph $G = (V, E)$ if $G - S$ is acyclic.



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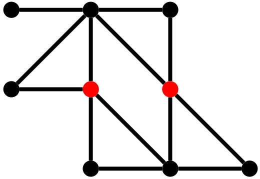

Exponential-time algorithms

- $O^*(2^n)$ trivial
- $O(1.8899^n)$ [Razgon 06]
- $O(1.7548^n)$ [Fomin, Gaspers, Pyatkin 06]
- $O(1.7356^n)$ [Xiao, Nagamoshi 13]
- $O(1.7347^n)$ [Fomin, Todinca, Villanger 15]

Parameterized algorithms

- $O^*((17k^4)!)$ [Bodlaender 94]
- $O^*((2k+1)^k)$ [Downey, Fellows 98]

$\vdots$

- $O^*(3.591^k)$ [Kociumaka, Pilipczuk 14]
- $O^*(3^k)$ (r) [Cygan 11]

Exponential-time algorithms via parameterized algorithms

Binomial coefficients

$$\arg\max_{0\le k\le n} \binom{n}{k} = n/2 \qquad \text{and} \qquad \binom{n}{n/2} = \Theta(2^n/\sqrt{n})$$

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Algorithm for Feedback Vertex Set

- Set $t = 0.60909 \cdot n$
- If $k \le t$, run $O^*(3^k)$ algorithm
- Else check all $\binom{n}{k}$ vertex subsets of size k

Running time: $O^*\left(\max\left(3^t, \binom{n}{t}\right)\right) = O^*(1.9526^n)$

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This approach gives algorithms faster than $O^*(2^n)$ for subset problems with a parameterized algorithm faster than $O^*(4^k)$.

Subset Problems

An *implicit set system* is a function Φ with:

- Input: instance $I \in \{0,1\}^*$, $|I| = N$
- Output: set system $(U_I, \mathcal{F}_I)$:
  - universe U_I, $|U_I| = n$
  - family $\mathcal{F}_I$ of subsets of U_I

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Φ-SUBSET

Input:	Instance I
Question:	Is $|\mathcal{F}_I| > 0$?

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Φ-Subset
Input: Instance I
Question: Is $|\mathcal{F}_I| > 0$?

Φ-Extension
Input: Instance I, a set $X \subseteq U_I$, and an integer k
Question: Does there exist a subset $S \subseteq (U_I \setminus X)$ such that $S \cup X \in \mathcal{F}_I$ and $|S| \leq k$?

Algorithm

Suppose Φ-EXTENSION has a $O^*(c^k)$ time algorithm B.

Algorithm for checking whether $\mathcal{F}_I$ contains a set of size k

- Set $t = \max\left(0, \frac{ck-n}{c-1}\right)$
- Uniformly at random select a subset $X \subseteq U_I$ of size t
- Run $B(I, X, k-t)$

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Running time: [Fomin, Gaspers, Lokshtanov, Saurabh 16]

$$O^*\left(\frac{\binom{n}{t}}{\binom{k}{t}} \cdot c^{k-t}\right) = O^*\left(2 - \frac{1}{c}\right)^n$$

Intuition

Brute-force randomized algorithm

- Pick k elements of the universe one-by-one.
- Suppose $\mathcal{F}_I$ contains a set of size k.

Success probability:

$$\frac{k}{n} \cdot \frac{k-1}{n-1} \cdot \ldots \cdot \underbrace{\frac{k-t}{n-t}}_{\substack{\| \\ \frac{1}{c}}} \cdot \ldots \cdot \frac{2}{n-(k-2)} \frac{1}{n-(k-1)} = \frac{1}{\binom{n}{k}}$$

Randomized Monotone Local Search

Theorem 13 ([Fomin, Gaspers, Lokshtanov, Saurabh 16])

If there exists a (randomized) algorithm for Φ-EXTENSION *with running time* $O^*(c^k)$ *then there exists a randomized algorithm for* Φ-SUBSET *with running time* $(2 - \frac{1}{c})^n \cdot N^{O(1)}$.

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Theorem 14 ([Fomin, Gaspers, Lokshtanov, Saurabh 16])

FEEDBACK VERTEX SET *has a randomized algorithm with running time* $O^*\left(\left(2 - \frac{1}{3}\right)^n\right) \subseteq O(1.6667^n)$.

Derandomization

Derandomization at the expense of a subexponential factor in the running time.

Theorem 15 ([Fomin, Gaspers, Lokshtanov, Saurabh 16])

If there exists an algorithm for Φ-EXTENSION *with running time* $O^*(c^k)$ *then there exists an algorithm for* Φ-SUBSET *with running time* $(2 - \frac{1}{c})^{n+o(n)} \cdot N^{O(1)}$.

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Theorem 16 ([Fomin, Gaspers, Lokshtanov, Saurabh 16])

FEEDBACK VERTEX SET *has an algorithm with running time* $O^*\left(\left(2 - \frac{1}{3.591}\right)^n\right) \subseteq O(1.7216^n)$.

Multivariate Subroutines

Theorem 17 ([Gaspers, Lee 17])

If there exists an algorithm for Φ-Extension *with running time* $O^*(b^{n-|X|} \cdot c^k)$ *then there exists an algorithm for* Φ-Subset *with running time* $(1 + b - \frac{1}{c})^{n+o(n)} \cdot N^{O(1)}$.

Theorem 18 ([Gaspers, Lee 17])

Feedback Vertex Set Extension *can be solved in time* $O(1.5422^{n-|X|} 1.2041^k)$.

Corollary 19 ([Gaspers, Lee 17])

Feedback Vertex Set *can be solved in time* $O(1.7117^n)$.

Further Reading

- Chapter 5, *Randomized methods in parameterized algorithms* by Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. Parameterized Algorithms. Springer, 2015.
- *Exact Algorithms via Monotone Local Search*, by Fedor V. Fomin, Serge Gaspers, Daniel Lokshtanov, Saket Saurabh. Proceedings of the 48th ACM Symposium on Theory of Computing (STOC 2016), ACM, pages 764-775, 2016.
- *Exact Algorithms via Multivariate Subroutines*, by Serge Gaspers and Edward J. Lee. Proceedings of the 44th International Colloquium on Automata, Languages and Programming (ICALP 2017), Track A. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, LIPIcs 80, pages 69:1–69:13, 2017.