1. Show that an irreflexive and transitive relation is asymmetric.

2. An equivalence relation on a set $A$ is any binary relation which is: a) reflexive; b) symmetric; and c) transitive.
   Show that for any fixed $m \in \mathbb{N}$, the relation $R_m \subseteq \mathbb{Z} \times \mathbb{Z}$ such that $xR_my$ iff $x - y = km$ for some $k \in \mathbb{Z}$, is an equivalence relation.
   Define $[n]_m = [n]_{R_m}$. Describe the equivalence class $[3]_0$, $[3]_1$, and $[3]_2$, $[3]_3$. In general, describe the equivalence classes $[n]_m$. Show that $[m]_4 \subseteq [m]_2$, for any $m \in \mathbb{Z}$. More generally, show that if for some $k, n, p \in \mathbb{Z}$, $n = kp$, then $[n]_m \subseteq [m]_p$.

3. Verify that for any finite (or indeed infinite) sets $A$ and $B$, the relation $A \simeq B$ iff $|A| = |B|$, where $|A|$ is the cardinality of $A$ (i.e., the number of elements in $A$) is an equivalence relation.

4. A partial order is any relation which is reflexive, antisymmetric, and transitive.
   Define the relation $| \subseteq \mathbb{N} \times \mathbb{N}$ by $x|y$ iff $x$ divides $y$ (or $x$ is a factor of $y$, or $y$ is a multiple of $x$). Show that $|$ is a partial order (i.e., that it is reflexive, antisymmetric, and transitive).

5. For a weak preference relation $\succsim$, verify the following:
   (a) If an agent’s preferences are consistent then $\sim$ is an equivalence relation.
   (b) The corresponding strict preference relation $\succ$ is a strict total order.
   (c) Strict preference satisfies an ‘indifference version’ of the trichotomy law; i.e., exactly one of the following holds between any $x, y \in A$: $x \succ y$ or $x \sim y$ or $y \succ x$.

6. Verify that the following properties hold from the axiomatisation of $\succsim$ given in lectures.
   - Strict preference properties:
     - if $x \succ y$, then it should be that $y \succ x$
     - if $x \succ y$ and $y \succ z$, then it should not be that $z \succ x$
   - Indifference properties:
     - if $x \sim y$, then $y \sim x$
     - if $x \sim y$ and $y \sim z$, then $x \sim z$
     - $x \sim x$ holds for any $x \in A$
   - Combined properties:
7. Let \([x]\) be an abbreviation for \([x]_\sim\), show that:

- (a) if \(x \sim y\), then \([x] = [y]\)
- (b) if \([x] \cap [y] \neq \emptyset\), then \([x] = [y]\)
- (c) if \(x \succ y\), then if \(a \in [x]\) and \(b \in [y]\), then \(a \succ b\)

8. Left the left-to-right edges in the Hasse diagram below represent \(\succ\).

\[
\begin{array}{ccc}
  & d & \\
 a & & c \\
 & f & c \\
 b & & g
\end{array}
\]

In terms of \(\succ\) what is the relationship between:

- (a) \(d\) and \(a\)
- (b) \(a\) and \(e\)
- (c) \(a\) and \(b\)
- (d) \(f\) and \(d\)

9. Consider the following preferences on the set \(A = \{a, b, c, d, e\}\):

\[
c \succ a \quad b \succ d \quad e \succ d \quad d \succ a \quad d \succ e \quad a \succ c
\]

- (a) What additional instances of \(\succ\) can be inferred from the axioms given in lectures?
- (b) Assume that any inference not present above, or inferred from them, is false. From the definition of \(\succ\) in terms of \(\succ\), what are the instances of \(\succ\)?
- (c) For an equivalence relation \((A, \sim)\), denote the set of all equivalence classes of \(A\) by \(A/\sim\). (Sometimes \(A/\sim\) is called the quotient class of \(A\).) List the indifferece classes in \(A/\sim\)?
- (d) Draw the Hasse diagram for \(\succ\).
- (e) Draw the Hasse diagram for \(\succ_I\): the preference relation on indifference classes.
- (f) Define an ordinal function \(V\) on the members of \(A/\sim\) \((i.e., V : A/\sim \rightarrow \mathbb{R})\) and hence, one on \(A\) \((v : A \rightarrow \mathbb{R})\).

10. Show that the weak preference ordering \(\succ_I\) on indifferece classes is antisymmetric.

11. Show that for the weak preference relation \(\succ_I\) on indifference classes:

- (a) for any \(X, Y \in A/\sim\), \(X \succ_I Y\) iff for every \(x \in X\) and \(y \in Y\), \(x \succ y\)
- (b) \(\succ_I\) is a weak total order
12. Show that for any ordinal value function $v$:

(a) $v(x) > v(y)$ iff $x > y$.
(b) $v(x) = v(y)$ iff $x \sim y$. 