12. Exponential Time Hypothesis

COMP6741: Parameterized and Exact Computation

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Contents

1 SAT and k-SAT

SAT

SAT	
Input:	A propositional formula F in conjunctive normal form (CNF)
Parameter:	$n = \|\mathsf{var}(F)\|$, the number of variables in F
Question:	Is there an assignment to $\mathsf{var}(F)$ satisfying all clauses of F?

k-SAT	
Input:	A CNF formula F where each clause has length at most k
Parameter:	$n = \|\mathsf{var}(F)\|$, the number of variables in F
Question:	Is there an assignment to $\mathsf{var}(F)$ satisfying all clauses of F?

Example:

$$(x_1 \lor x_2) \land (\neg x_2 \lor x_3 \lor \neg x_4) \land (x_1 \lor x_4) \land (\neg x_1 \lor \neg x_3 \lor \neg x_4)$$

Algorithms for SAT

- Brute-force: $O^*(2^n)$
- ... after > 50 years of SAT solving (SAT association, SAT conference, JSAT journal, annual SAT competitions, ...)
- fastest known algorithm for SAT: $O^*(2^{n\cdot(1-1/O(\log m/n))})$, where m is the number of clauses [Calabro, Impagliazzo, Paturi, 2006] [Dantsin, Hirsch, 2009]
- However: no $O^*(1.9999^n)$ time algorithm is known

- fastest known algorithms for 3-SAT: $O^*(1.3303^n)$ deterministic [Makino, Tamaki, Yamamoto, 2013] and $O^*(1.3071^n)$ randomized [Hertli, 2014]
- Could it be that 3-SAT cannot be solved in $2^{o(n)}$ time?
- Could it be that SAT cannot be solved in $O^*((2-\epsilon)^n)$ time for any $\epsilon > 0$?

2 Subexponential time algorithms

NP-hard problems in subexponential time?

- Are there any NP-hard problems that can be solved in $2^{o(n)}$ time?
- **Yes**. For example, Independent Set is NP-comlpete even when the input graph is planar (can be drawn in the plane without edge crossings). Planar graphs have treewidth $O(\sqrt{n})$ and tree decompositions of that width can be found in polynomial time ("Planar separator theorem" [Lipton, Tarjan, 1979]). Using a tree decomposition based algorithm, Independent Set can be solved in $2^{O(\sqrt{n})}$ time on planar graphs.

3 ETH and SETH

Definition 1. For each $k \geq 3$, define δ_k to be the infinimum[1] of the set of constants c such that k-SAT can be solved in $O^*(2^{c \cdot n})$ time.

Conjecture 2 (Exponential Time Hyphothesis (ETH))**.** $\delta_3 > 0$.

Conjecture 3 (Strong Exponential Time Hyphothesis (SETH))**.** $\lim_{k\to\infty} \delta_k = 1$.

Notes: (1) ETH $\Rightarrow$ 3-SAT cannot be solved in $2^{o(n)}$ time. SETH $\Rightarrow$ SAT cannot be solved in $O^*((2-\epsilon)^n)$ time for any $\epsilon > 0$.

4 Algorithmic lower bounds based on ETH

- Suppose ETH is true
- Can we infer lower bounds on the running time needed to solve other problems?
- Suppose there is a polynomial-time reduction from 3-SAT to a graph problem Π, which constructs an equivalent instance where the number of vertices of the output graph equals the number of variables of the input formula, $|V| = |\mathsf{var}(F)|$.
- Using the reduction, we can conclude that, if Π has an $O^*(2^{o(|V|)})$ time algorithm, then 3-SAT has an $O^*(2^{o(|\mathsf{var}(F)|)})$ time algorithm, contradicting ETH.
- Therefore, we conclude that Π has no $O^*(2^{o(|V|)})$ time algorithm unless ETH fails.

Sparsification Lemma
Issue: Many reductions from 3-SAT create a number of vertices / variables / elements that are related to the number of *clauses* of the 3-SAT instance.

Theorem 4 (Sparsification Lemma, [Impagliazzo, Paturi, Zane, 2001])**.** *For each $\varepsilon > 0$ and positive integer k, there is a $O^*(2^{\varepsilon \cdot n})$ time algorithm that takes as input a k-CNF formula F with n variables and outputs an equivalent formula $F' = \bigvee_{i=1}^t F_i$ that is a disjunction of $t \leq 2^{\varepsilon n}$ formulas F_i with $\mathsf{var}(F_i) = \mathsf{var}(F)$ and $|\mathsf{cla}(F_i)| = O(n)$.*

[1] The infinimum of a set of numbers is the largest number that is smaller or equal to each number in the set. E.g., the infinimum of $\{\varepsilon \in \mathbb{R} : \varepsilon > 0\}$ is 0.

3-SAT with a linear number of clauses

Corollary 5. *ETH ⇒ 3-SAT cannot be solved in $O^*(2^{o(n+m)})$ time where m denotes the number of clauses of F.*

Observation: Let A, B be parameterized problems and f, g be non-decreasing functions. Suppose there is a polynomial-parameter transformation from A to B such that if the parameter of an instance of A is k, then the parameter of the constructed instance of B is at most $g(k)$. Then an $O^*(2^{o(f(k))})$ time algorithm for B implies an $O^*(2^{o(f(g(k)))})$ time algorithm for A.

More general reductions are possible

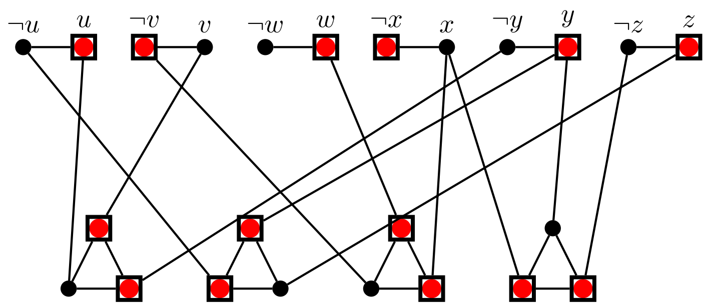
Definition 6 (SERF-reduction)**.** A *SubExponential Reduction Family* from a parameterized problem A to a parameterized problem B is a family of *Turing reductions* from A to B (i.e., an algorithm for A, making queries to an *oracle* for B that solves any instance for B in constant time) for each $\varepsilon > 0$ such that

- for every instance I for A with parameter k, the running time is $O^*(2^{\varepsilon k})$, and
- for every query I' to B with parameter k', we have that $k' \in O(k)$ and $|I'| = |I|^{O(1)}$.

Note: If A is SERF-reducible to B and A has no $2^{o(k)}$ time algorithm, then B has no $2^{o(k')}$ time algorithm.

Vertex Cover has no subexponential algorithm

Polynomial-parameter transformation from 3-SAT.
For simplicity, assume all clauses have length 3.
3-CNF Formula $F = (u \vee v \vee \neg y) \wedge (\neg u \vee y \vee z) \wedge (\neg v \vee w \vee x) \wedge (x \vee y \vee \neg z)$



For a 3-CNF formula with n variables and m clauses, we create a Vertex Cover instance with $|V| = 2n + 3m$, $|E| = n + 6m$, and $k = n + 2m$.

Theorem 7. *ETH ⇒* Vertex Cover *has no $2^{o(|V|)}$ time algorithm.*

Theorem 8. *ETH ⇒* Vertex Cover *has no $2^{o(|E|)}$ time algorithm.*

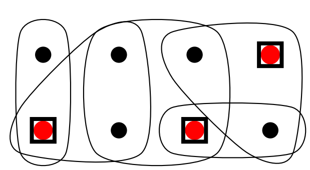
Theorem 9. *ETH ⇒* Vertex Cover *has no $2^{o(k)}$ time algorithm.*

5 Algorithmic lower bounds based on SETH

Hitting Set

Recall: A *hitting set* of a set system $\mathcal{S} = (V, H)$ is a subset X of V such that X contains at least one element of each set in H, i.e., $X \cap Y \neq \emptyset$ for each $Y \in H$.

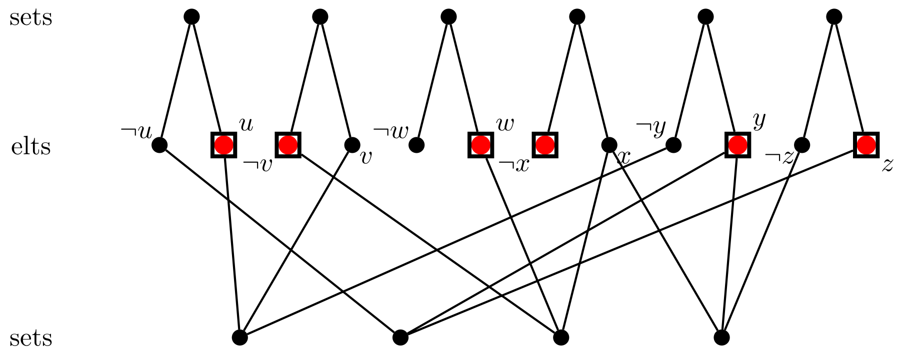
elts-Hitting Set	
Input:	A set system $\mathcal{S} = (V, H)$ and an integer k
Parameter:	$n = \|V\|$
Question:	Does $\mathcal{S}$ have a hitting set of size at most k?



SETH-lower bound for Hitting Set

CNF Formula $F = (u \vee v \vee \neg y) \wedge (\neg u \vee y \vee z) \wedge (\neg v \vee w \vee x) \wedge (x \vee y \vee \neg z)$

Inidence graph of equivalent Hitting Set instance:



For a CNF formula with n variables and m clauses, we create a Hitting Set instance with $|V| = 2n$ and $k = n$.

Theorem 10. *SETH* $\Rightarrow$ Hitting Set *has no* $O^*((2-\varepsilon)^{|V|/2})$ *time algorithm for any* $\varepsilon > 0$.

Note: With a more ingenious reduction, one can show that Hitting Set has no $O^*((2-\varepsilon)^{|V|})$ time algorithm for any $\varepsilon > 0$ under SETH.

Exercise

A *dominating set* of a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that $N_G[S] = V$.

vertex-Dominating Set	
Input:	A graph $G = (V, E)$ and an integer k
Parameter:	$n = \|V\|$
Question:	Does G have a dominating set of size at most k?

- Prove that ETH $\Rightarrow$ vertex-Dominating Set has no $2^{o(n)}$ time algorithm.

Solution idea

Reduce from 3-SAT. For each $x \in \mathsf{var}(F)$, create a triangle with vertices x, $\neg x$ and d_x. For each $c \in \mathsf{cla}(F)$, create a vertex c adjacent to all the vertices whose corresponding literals are contained in the clause c.

6 Further Reading

- Chapter 14, *Lower bounds based on the Exponential-Time Hypothesis* in Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, MichałPilipczuk, and Saket Saurabh. Parameterized Algorithms. Springer, 2015.

- Section 11.3, *Subexponential Algorithms and ETH* in Fedor V. Fomin and Dieter Kratsch. Exact Exponential Algorithms. Springer, 2010.

- Section 29.5, *The Sparsification Lemma* in Rodney G. Downey and Michael R. Fellows. Fundamentals of Parameterized Complexity. Springer, 2013.