Assignment 1
COMP6741: Parameterized and Exact Computation

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Assignment 1 is an individual assignment. For the solutions to this assignment, you may rely on all theorems, lemmas, and results from the lecture notes. If any other works (articles, Wikipedia entries, lecture notes from other courses, etc.) inspired your solutions, please cite them and give a list of references at the end.

If you have questions about this assignment, please post them to the Forum.

Due date. This assignment is due on Wednesday, 16 August 2017, at 23.59 AEST. Submitting x days after the deadline, with $x > 0$, reduces the grade by $20 \cdot x$ per cent.

How to submit. Submit a PDF with your solutions using the command

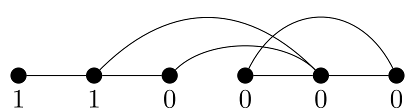
give cs6741 a1 <mysolution.pdf>

from the CSE network, or using the new WebCMS3 frontend for `give`. The first page of the PDF must contain your name and student number.

Exercise 1. We consider the problem of cleaning a graph $G = (V, E)$ using a smallest number of brushes. In a cleaning process, brushes are initially placed on vertices. A vertex can be cleaned if its number of brushes is at least its degree. When a vertex u is cleaned, exactly one brush is sent along each of its incident edges to a neighboring vertex. The vertex is then deleted, along with all its incident edges and any brushes that still remain on this vertex. The brush number of G is the smallest number of brushes needed so that all the vertices can be cleaned.

GRAPH-CLEANING	
Input:	Graph $G = (V, E)$, integer k
Question:	Is the brush number of G at most k?

Example: To clean the following graph, two brushes are sufficient.



- Show that the brush number of a graph is at least the minimum degree of the graph. [5 points]
- Show that the brush number of a graph G is at least $\lceil \Delta(G)/2 \rceil$, where $\Delta(G)$ is the maximum degree of the graph. [5 points]
- Reformulate the GRAPH-CLEANING problem as a permutation problem. In other words, complete the following sentence: The brush number of a graph $G = (V, E)$ is the smallest integer k such that there exists a permutation $\pi = (v_1, v_2, \ldots, v_n)$ of the vertex set V such that ... Feel free to define new concepts in doing so. [10 points]
- Design an algorithm that solves GRAPH-CLEANING in time $O^*(2^n)$, where n is the number of vertices of G. [30 points]

Exercise 2. Consider the SPECIFIED 5-SET COVER problem (S5SC). The input is a set system $(U, \mathcal{S})$ of rank 5, that is a finite set U and a set $\mathcal{S}$ that contains subsets of U of size at most 5. Moreover, each $S \in \mathcal{S}$ is associated an integer r_S. The question is whether one can select elements from U such that for each set $S \in \mathcal{S}$, exactly r_S elements are selected.

SPECIFIED 5-SET COVER (S5SC)	
Input:	A set system $(U, \mathcal{S})$ where each set $S \in \mathcal{S}$ is a subset of U and has size at most 5, and an integer $r_S \in \{0, \dots, 5\}$ for each set $S \in \mathcal{S}$.
Question:	Is there a set $X \subseteq U$ such that $|X \cap S| = r_S$ for each $S \in \mathcal{S}$?

Example:

$$\big((U = \{a, b, c, d, e\}, S = \{\{a, b\}, \{a, b, c, d\}, \{b, c, d, e\}, \{c, e\}\}),\\ r_{\{a,b\}} = 1, r_{\{a,b,c,d\}} = 2, r_{\{b,c,d,e\}} = 1, r_{\{c,e\}} = 0\big)$$

is a YES-instance for S5SC, certified by the set $X = \{a, d\}$.

- Design an algorithm for S5SC and analyze its running time with respect to $n = |U|$. [50 points]

  The number of awarded points for a correct algorithm and analysis depends on the running time: $O^*(2^n)$ gives 20 points, $O^*(1.8^n)$ gives 25 points, $O^*(1.62^n)$ gives 30 points, $O^*(1.59^n)$ gives 45 points, and $O^*(1.58^n)$ gives 50 points, where n is the number of elements in U.