# 8a. Randomized Algorithms

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## Contents



## 1 Introduction

### Randomized Algorithms

- Turing machines do not inherently have access to randomness.
- Assume algorithm has also access to a stream of *random bits* drawn uniformly at random.
- With $r$ random bits, the probability space is the set of all $2^r$ possible strings of random bits (with uniform distribution).

### Las Vegas algorithms

**Definition 1.** A *Las Vegas algorithm* is a randomized algorithm whose output is always correct.

Randomness is used to upper bound the expected running time of the algorithm.

**Example**
Quicksort with random choice of pivot.

### Monte Carlo algorithms

**Definition 2.**
- A *Monte Carlo algorithm* is an algorithm whose output is incorrect with probability at most $p$, $0 < p < 1$.
- A Monte Carlo has *one sided* error if its output is incorrect only on Yes-instances or on No-instances, but not both.
- A one-sided error Monte Carlo algorithm with *false negatives* answers No for every No-instance, and answers Yes on Yes-instances with probability $p \in (0, 1)$. We say that $p$ is the *success probability* of the algorithm.

**Boosting success probability**
Suppose $A$ is a one-sided Monte Carlo algorithm with false negatives with success probability $p$. How can we use $A$ to design a new one-sided Monte Carlo algorithm with success probability $p^* > p$?

Let $t = -\frac{\ln(1-p^*)}{p}$ and run the algorithm $t$ times. Return YES if at least one run of the algorithm returned YES, and NO otherwise. Failure probability is

$$(1-p)^t \leq (e^{-p})^t = e^{-p \cdot t} = e^{\ln(1-p^*)} = 1 - p^*$$

via the inequality $1 - x \leq e^{-x}$.

**Definition 3.** A *randomized algorithm* is a one-sided Monte Carlo algorithm with *constant* success probability.

**Amplification**

**Theorem 4.** *If a one-sided error Monte Carlo algorithm has success probability at least p, then repeating it independently $\lceil \frac{1}{p} \rceil$ times gives constant success probability.*

In particular if we have a polynomial-time one-sided error Monte Carlo algorithm with success probability $p = \frac{1}{f(k)}$ for some computable function $f$, then we get a randomized FPT algorithm with running time $O^*(f(k))$.

## 2 Vertex Cover

For a graph $G = (V, E)$ a *vertex cover* $X \subseteq V$ is a set of vertices such that every edge is adjacent to a vertex in $X$.

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|---|---|
| VERTEX COVER | |
| Input: | Graph $G$, integer $k$ |
| Parameter: | $k$ |
| Question: | Does $G$ have a vertex cover of size $k$? |

**Warm-up:** design a randomized algorithm with running time $O^*(2^k)$.

*Algorithm rvc*$(G = (V, E), k)$

$S \leftarrow \emptyset$
**while** $k > 0$ and $E \neq \emptyset$ **do**
  Select an edge $uv \in E$ uniformly at random
  Select an endpoint $w \in \{u, v\}$ uniformly at random
  $S \leftarrow S \cup \{w\}$
  $G \leftarrow G - w$
  $k \leftarrow k - 1$
**if** $S$ is a vertex cover of $G$ **then**
  **return** YES
**else**
  **return** NO

**Success probability**

- Let $C$ be a minimal vertex cover of $G$ of size $k$
- What is the probability that Algorithm rvc returns $C$?
- When it selects an edge $uv \in E$, we have that $\{u, v\} \cap C \neq \emptyset$
- When it selects a random endpoint $w \in \{u, v\}$, we have that $w \in C$ with probability $\geq 1/2$
- It finds $C$ with probability at least $1/2^k$

**Theorem 5.** VERTEX COVER *has a randomized algorithm with running time $O^*(2^k)$.*

*Proof.*
- If $G$ has vertex cover number at most $k$, then Algorithm rvc finds one with probability at least $\frac{1}{2^k}$.
- Applying Theorem 4 gives a randomized FPT running time of $O^*(2^k)$.

□

## 3 Feedback Vertex Set

A *feedback vertex set* of a multigraph $G = (V, E)$ is a set of vertices $S \subset V$ such that $G - S$ is acyclic.

| | |
|---|---|
| Feedback Vertex Set | |
| Input: | Multigraph $G$, integer $k$ |
| Parameter: | $k$ |
| Question: | Does $G$ have a feedback vertex of size $k$? |

Recall the following simplification rules for Feedback Vertex Set.

### Simplification Rules

1. Loop: If loop at vertex $v$, remove $v$ and decrease $k$ by 1
2. Multiedge: Reduce the multiplicity of each edge with multiplicity $\geq 3$ to 2.
3. Degree-1: If $v$ has degree at most 1 then remove $v$.
4. Degree-2: If $v$ has degree 2 with neighbors $u, w$ then delete 2 edges $uv, vw$ and replace with new edge $uw$.

### The solution is incident to a constant fraction of the edges

**Lemma 6.** *Let $G$ be a multigraph with minimum degree at least 3. Then, for every feedback vertex set $X$ of $G$, at least $1/3$ of the edges have at least one endpoint in $X$.*

*Proof.* Denote by $n$ and $m$ the number of vertices and edges of $G$, respectively. Since $\delta(G) \geq 3$, we have that $m \geq 3n/2$. Let $F := G - X$. Since $F$ has at most $n - 1$ edges, at least $\frac{1}{3}$ of the edges have an endpoint in $X$. □

### Randomized Algorithm

**Theorem 7.** Feedback Vertex Set *has a randomized algorithm with running time $O^*(6^k)$.*

We prove the theorem using the following algorithm.

- $S \leftarrow \emptyset$
- Do $k$ times: Apply simplification rules; add a random endpoint of a random edge to $S$.
- If $S$ is a feedback vertex set, return Yes, otherwise return No.

*Proof.*
- We need to show: each time the algorithm adds a vertex $v$ to $S$, if $(G - S, k - |S|)$ is a Yes-instance, then with probability at least $1/6$, the instance $(G - (S \cup \{v\}), k - |S| - 1)$ is also a Yes-instance. Then, by induction, we can conclude that with probability $1/(6^k)$, the algorithm finds a feedback vertex set of size at most $k$ if it is given a Yes-instance.
- Assume $(G - S, k - |S|)$ is a Yes-instance.
- Lemma 6 implies that with probability at least $1/3$, a randomly chosen edge $uv$ has at least one endpoint in some feedback vertex set of size $k - |S|$.
- So, with probability at least $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$, a randomly chosen endpoint of $uv$ belongs some feedback vertex set of size $\leq k - |S|$.
- Applying Theorem 4 gives a randomized FPT running time of $O^*(6^k)$.

□

### Improved analysis

**Lemma 8.** *Let $G$ be a multigraph with minimum degree at least 3. For every feedback vertex set $X$, at least $1/2$ of the edges of $G$ have at least one endpoint in $X$.*

**Note:** For a feedback vertex set $X$, consider the forest $F := G - X$. The statement is equivalent to:

$$|E(G) \setminus E(F)| \geq |E(F)|$$

Let $J \subseteq E(G)$ denote the edges with one endpoint in $X$, and the other in $V(F)$. We will show the stronger result:

$$|J| \geq |V(F)|$$

*Proof.*

- Let $V_{\leq 1}, V_2, V_{\geq 3}$ be the set of vertices that have degree at most 1, exactly 2, and at least 3, respectively, in $F$.
- Since $\delta(G) \geq 3$, each vertex in $V_{\leq 1}$ contributes at least 2 edges to $J$, and each vertex in $V_2$ contributes at least 1 edge to $J$.
- We show that $|V_{\geq 3}| \leq |V_{\leq 1}|$ by induction on $|V(F)|$.
  - Trivially true for forests with at most 1 vertex.
  - Assume true for forests with at most $n-1$ vertices.
  - For any forest on $n$ vertices, consider removing a leaf (which must always exist) to obtain $F'$ with the vertex partition $(V'_{\leq 1}, V'_2, V'_{\geq 3})$. If $|V_{\geq 3}| = |V'_{\geq 3}|$, then we have that $|V_{\geq 3}| = |V'_{\geq 3}| \leq |V'_{\leq 1}| \leq |V_{\geq 1}|$. Otherwise, $|V_{\geq 3}| = |V'_{\geq 3}| + 1 \leq |V'_{\leq 1}| + 1 = |V_{\leq 1}|$.
- We conclude that:

$$|E(G) \setminus E(F)| \geq |J| \geq 2|V_{\leq 1}| + |V_2| \geq |V_{\leq 1}| + |V_2| + |V_{\geq 3}| = |V(F)|$$

□

### Improved Randomized Algorithm

**Theorem 9.** Feedback Vertex Set *has a randomized algorithm with running time $O^*(4^k)$.*

**Note**
This algorithmic method is applicable whenever the vertex set we seek is incident to a constant fraction of the edges.

# 4 Color Coding

### Longest Path

| Longest Path | |
|---|---|
| Input: | Graph $G$, integer $k$ |
| Parameter: | $k$ |
| Question: | Does $G$ have a path on $k$ vertices as a subgraph? |

**NP-complete**
To show that Longest Path is NP-hard, reduce from Hamiltonian Path by setting $k = n$ and leaving the graph unchanged.

**Color Coding**
**Notation:** $[k] = \{1, 2, \ldots, k\}$

**Lemma 10.** *Let $U$ be a set of size $n$, and let $X \subseteq U$ be a subset of size $k$. Let $\chi : U \to [k]$ be a coloring of the elements of $U$, chosen uniformly at random. Then the probability that the elements of $X$ are colored with pairwise distinct colors is at least $e^{-k}$.*

*Proof.* There are $k^n$ possible colorings $\chi$ and $k!k^{n-k}$ of them are injective on $X$. Using the inequality

$$k! > (k/e)^k,$$

the lemma follows since

$$\frac{k! \cdot k^{n-k}}{k^n} > \frac{k^k \cdot k^{n-k}}{e^k \cdot k^n} = e^{-k}.$$

□

**Colorful Path**
A path is *colorful* if all vertices of the path are colored with pairwise distinct colors.

**Lemma 11.** *Let $G$ be an undirected graph, and let $\chi : V(G) \to [k]$ be a coloring of its vertices with $k$ colors. There is an algorithm that checks in time $O^*(2^k)$ whether $G$ contains a colorful path on $k$ vertices.*

*Proof.* Partition $V(G)$ into $V_1, ..., V_k$ subsets such that vertices in $V_i$ are colored $i$.
Apply dynamic programming on nonempty $S \subseteq \{1, ..., k\}$. For $u \in \bigcup_{i \in S} V_i$ let $P(S, u) = true$ if there is a colorful path with colors from $S$ and $u$ as an endpoint. We have the following:

- For $|S| = 1$, $P(S, u) = true$ for $u \in V(G)$ iff $S = \{\chi(u)\}$.
- For $|S| > 1$

$$P(S,u) = \begin{cases} \bigvee_{uv \in E(G)} P(S \backslash \{\chi(u)\}, v) & \text{if } \chi(u) \in S \\ false & \text{otherwise} \end{cases}$$

All values of $P$ can be computed in $O^*(2^k)$ time and there exists a colorful $k$-path iff $P([k], v)$ is true for some vertex $v \in V(G)$. □

**Theorem 12.** Longest Path *has a randomized algorithm with running time $O^*((2 \cdot e)^k)$.*

**Note**
This algorithmic method is applicable whenever we seek a vertex set $S$ of size $f(k)$ such that $G[S]$ has constant treewidth.

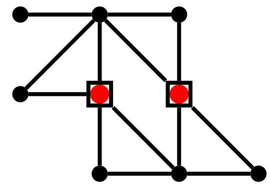
# 5 Monotone Local Search

| Exponential-time algorithms | Parameterized algorithms |
|---|---|
| Algorithms for NP-hard problems | Algorithms for NP-hard problems |
| Beat brute-force & improve | Use a parameter $k$ (often $k$ is the solution size) |
| Running time measured in the size of the universe $n$ | Algorithms with running time $f(k) \cdot n^c$ |
| $O(2^n \cdot n)$, $O(1.5086^n)$, $O(1.0892^n)$ | $k^k n^{O(1)}$, $5^k n^{O(1)}$, $O(1.2738^k + kn)$ |

Can we use Parameterized algorithms to design fast Exponential-time algorithms?

**Example: Feedback Vertex Set**

$S \subseteq V$ is a *feedback vertex set* in a graph $G = (V, E)$ if $G - S$ is acyclic.



| | |
|---|---|
| FEEDBACK VERTEX SET | |
| Input: | Graph $G = (V, E)$, integer $k$ |
| Parameter: | $k$ |
| Question: | Does $G$ have a feedback vertex set of size at most $k$? |

### Exponential-time algorithms

- $O^*(2^n)$ trivial
- $O(1.7548^n)$ [Fom+08]
- $O(1.7347^n)$ [FV10]
- $O(1.7266^n)$ [XN15]

### Parameterized algorithms

- $O^*((17k^4)!)$ [Bod94]
- $O^*((2k+1)^k)$ [DF99]

$\vdots$

- $O^*(3.460^k)$ deterministic [IK19]
- $O^*(2.7^k)$ randomized [LN19]

**Exponential-time algorithms via parameterized algorithms**

**Binomial coefficients**

$$\arg\max_{0 \le k \le n} \binom{n}{k} = n/2 \qquad \text{and} \qquad \binom{n}{n/2} = \Theta(2^n/\sqrt{n})$$

**Algorithm for Feedback Vertex Set**

- Set $t = 0.60909 \cdot n$
- If $k \le t$, run $O^*(3^k)$ algorithm
- Else check all $\binom{n}{k}$ vertex subsets of size $k$

Running time: $O^*\left(\max\left(3^t, \binom{n}{t}\right)\right) = O^*(1.9526^n)$

This approach gives algorithms faster than $O^*(2^n)$ for subset problems with a parameterized algorithm faster than $O^*(4^k)$.

**Subset Problems**

An *implicit set system* is a function $\Phi$ with:

- Input: instance $I \in \{0,1\}^*$, $|I| = N$
- Output: set system $(U_I, \mathcal{F}_I)$:
  - universe $U_I$, $|U_I| = n$
  - family $\mathcal{F}_I$ of subsets of $U_I$

| | |
|---|---|
| $\Phi$-SUBSET | |
| Input: | Instance I |
| Question: | Is $\lvert\mathcal{F}_I\rvert > 0$? |

| | |
|---|---|
| $\Phi$-EXTENSION | |
| Input: | Instance $I$, a set $X \subseteq U_I$, and an integer $k$ |
| Question: | Does there exist a subset $S \subseteq (U_I \backslash X)$ such that $S \cup X \in \mathcal{F}_I$ and $\lvert S\rvert \le k$? |

**Algorithm**

Suppose $\Phi$-Extension has a $O^*(c^k)$ time algorithm $B$.

**Algorithm for checking whether $\mathcal{F}_I$ contains a set of size $k$**

- Set $t = \max\left(0, \frac{ck-n}{c-1}\right)$
- Uniformly at random select a subset $X \subseteq U_I$ of size $t$
- Run $B(I, X, k-t)$

Running time: [Fom+19]

$$O^*\left(\frac{\binom{n}{t}}{\binom{k}{t}} \cdot c^{k-t}\right) = O^*\left(2 - \frac{1}{c}\right)^n$$

**Intuition**

**Brute-force randomized algorithm**

- Pick $k$ elements of the universe one-by-one.
- Suppose $\mathcal{F}_I$ contains a set of size $k$.

Success probability:

$$\frac{k}{n} \cdot \frac{k-1}{n-1} \cdot \ldots \cdot \underbrace{\frac{k-t}{n-t}}_{= \frac{1}{c}} \cdot \ldots \cdot \frac{2}{n-(k-2)} \frac{1}{n-(k-1)} = \frac{1}{\binom{n}{k}}$$

**Theorem 13** ([Fom+19]). *If there exists a (randomized) algorithm for* $\Phi$-Extension *with running time* $O^*(c^k)$ *then there exists a randomized algorithm for* $\Phi$-Subset *with running time* $(2-\frac{1}{c})^n \cdot N^{O(1)}$.

**Theorem 14** ([Fom+19]). Feedback Vertex Set *has a randomized algorithm with running time* $O^*\left(\left(2-\frac{1}{2.7}\right)^n\right) \subseteq O(1.6297^n)$.

**Derandomization**

Derandomization at the expense of a subexponential factor in the running time.

**Theorem 15** ([Fom+19]). *If there exists an algorithm for* $\Phi$-Extension *with running time* $O^*(c^k)$ *then there exists an algorithm for* $\Phi$-Subset *with running time* $(2-\frac{1}{c})^{n+o(n)} \cdot N^{O(1)}$.

**Theorem 16** ([Fom+19]). Feedback Vertex Set *has an algorithm with running time* $O^*\left(\left(2-\frac{1}{3.460}\right)^n\right) \subseteq O(1.7110^n)$.

**Further Reading**

- Chapter 5, *Randomized methods in parameterized algorithms* by [Cyg+15]
- *Exact Algorithms via Monotone Local Search* [Fom+19]

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