COMP9020

Foundations of Computer Science

Lecture 5: Relations

Relations and Functions

Relations are an abstraction used to capture the idea that the objects from certain domains (often the same domain for several objects) are *related*. These objects may

- influence one another (each other for binary relations; self(?) for unary)
- share some common properties
- correspond to each other precisely when some constraints are satisfied

Functions capture the idea of transforming *inputs* into *outputs*.

In general, functions and relations formalise the concept of interaction among objects from various domains; however, there must be a specified domain for each type of objects.

Applications in Computer Science

- Relations are the building blocks of nearly all Computer Science structures
- Databases are collections of relations
- Any ordering is a relation
- Common data structures (e.g. graphs) are relations
- Functions/procedures/programs compute relations between their input and output

Applications in Computer Science

Many binary relations (i.e. relationships between two entities) that appear in CS fall into two broad categories:

Equivalence relations (generalizing "equality"):

- Programs that exhibit the same behaviour
- Logically equivalent statements
- The `.equals()` method in Java

Partial orders (generalizing "less than or equal to"):

- Object inheritance
- Simulation
- Requirement specifications
- The `.compareTo()` method in Java

Outline

Definition and Examples

Binary Relations

Properties of Binary Relations

Functions

Feedback

Relations

Definition

An **n-ary relation** is a subset of the cartesian product of n sets.

$$R \subseteq S_1 \times S_2 \times \ldots \times S_n$$

To show tuples related by R we write:

$$(x_1, x_2, \ldots, x_n) \in R \quad \text{or} \quad R(x_1, x_2, \ldots, x_n)$$

If $n = 2$ we have a **binary** relation $R \subseteq S \times T$ and to show pairs related by R we write:

$$(x, y) \in R \quad \text{or} \quad R(x, y) \quad \text{or} \quad xRy$$

$\mathcal{U} = S_1 \times S_2 \times \ldots \times S_n$ is the **domain** of R, and we say R is a **relation on** $\mathcal{U}$ (or **on** S if $S_1 = \cdots = S_n = S$ and n is clear).

Examples

Examples

- Equality: $=$
- Inequality: $\leq$, $\geq$, $<$, $>$, $\neq$
- Divides relation: $|$
- Element of: $\in$
- Subset, superset: $\subseteq$, $\subset$, $\supseteq$, $\supset$
- Congruence modulo n: $m =_{(n)} p$

Database Examples

Example (Course enrolments)

S = set of CSE students
(S can be a subset of the set of all students)
C = set of CSE courses
(likewise)
E = enrolments = $\{ (s, c) : s \text{ takes } c \}$

$$E \subseteq S \times C$$

In practice, almost always there are various 'onto' (nonemptiness) and 1–1 (uniqueness) constraints on database relations.

Example (Class schedule)

C = CSE courses
T = starting time (hour & day)
R = lecture rooms
S = schedule =

$$\{ (c, t, r) : c \text{ is at } t \text{ in } r \} \subseteq C \times T \times R$$

Example (sport stats)

$$R \subseteq \text{competitions} \times \text{results} \times \text{years} \times \text{athletes}$$

Defining Relations

Just as with sets R can be defined by

- explicit enumeration of interrelated k-tuples (ordered pairs in case of binary relations);
- properties that identify relevant tuples within the entire $S_1 \times S_2 \times \ldots \times S_k$;
- construction from other relations (e.g. union, intersection, complement etc).

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Binary relations

A **binary relation between** S **and** T is a subset of $S \times T$: i.e. a set of ordered pairs.
Also: over S and T; from S to T; on S (if $S = T$).

Example (Special (Trivial) Relations)

- **Identity**: (diagonal, equality) $I = \{ (x, x) : x \in S \}$
- **Empty**: $\emptyset$
- **Universal**: $U = S \times S$

Defining binary relations: Set-based definitions

Defining a relation $R \subseteq S \times T$:

- Explicitly listing tuples: e.g. $\{(1,1),(2,3),(3,2)\}$
- Set comprehension: $\{(x,y) \in [1,3] \times [1,3] : 5|xy-1\}$
- Construction from other relations:
  $\{(1,1)\} \cup \{(2,3)\} \cup \{(2,3)\}^{\leftarrow}$

Defining binary relations: Matrix representation

Defining a relation $R \subseteq S \times T$:
Rows enumerated by elements of S, columns by elements of T:

Examples

- The relation $\{(1,1),(2,3),(3,2)\} \subseteq [1,3] \times [1,3]$:

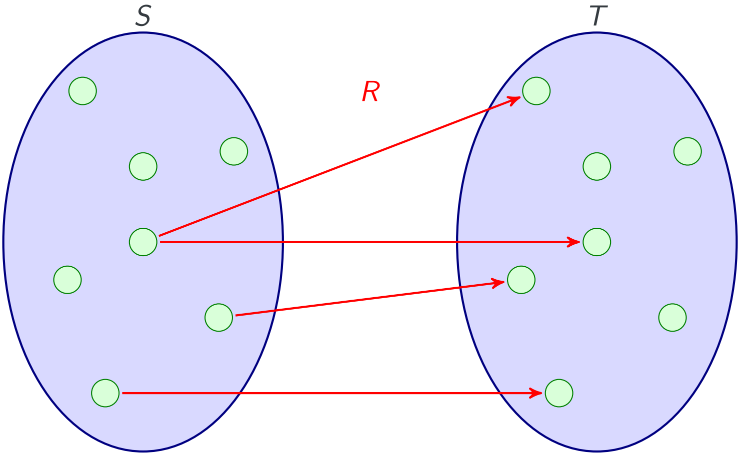
$$\begin{bmatrix} \bullet & \circ & \circ \\ \circ & \circ & \bullet \\ \circ & \bullet & \circ \end{bmatrix}$$

- The relation $\{(1,1),(1,2),(1,3),(1,4),(2,2),(3,2)\} \subseteq [1,3] \times [1,4]$:

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \circ & \circ \\ \circ & \bullet & \circ & \circ \end{bmatrix}$$

Defining binary relations: Graphical representation

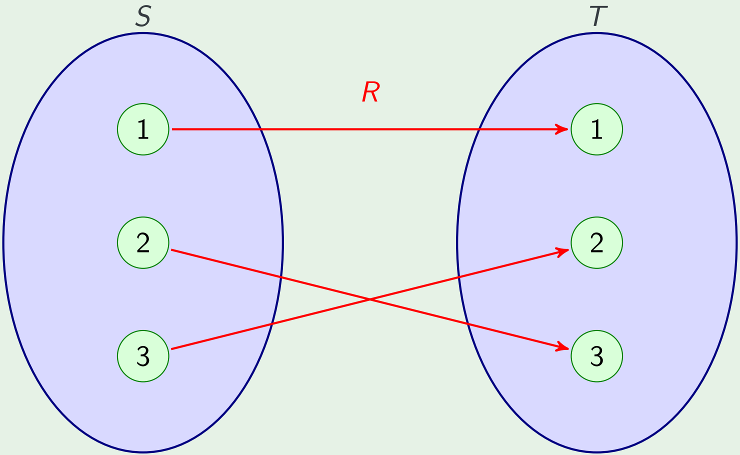
Defining a relation $R \subseteq S \times T$:



Defining binary relations: Graphical representation

Example

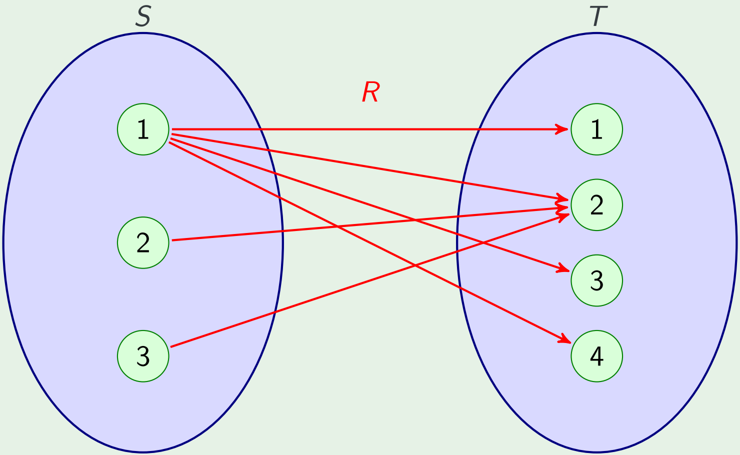
$R = \{(1,1),(2,3),(3,2)\} \subseteq [1,3] \times [1,3]$:



Defining binary relations: Graphical representation

Example

$\{(1,1),(1,2),(1,3),(1,4),(2,2),(3,2)\} \subseteq [1,3] \times [1,4]$:



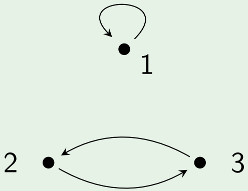
Defining binary relations: Graph representation

If $S = T$ we can define $R \subseteq S \times S$ as a **directed graph** (week 5).

- Nodes: Elements of S
- Edges: Elements of R

Example

$R = \{(1,1),(2,3),(3,2)\} \subseteq [1,3] \times [1,3]$:



Operations for binary relations

Relations are sets, so the standard set operations ($\cap$, $\cup$, $\setminus$, $\oplus$, etc) can be used to build new relations.

Two operations that apply to binary relations uniquely:

- **Converse**: If $R \subseteq S \times T$ is a relation, then $R^{\leftarrow} \subseteq T \times S$:

$$R^{\leftarrow} \stackrel{\text{def}}{=} \{(t, s) \in T \times S \ : \ (s, t) \in R\}$$

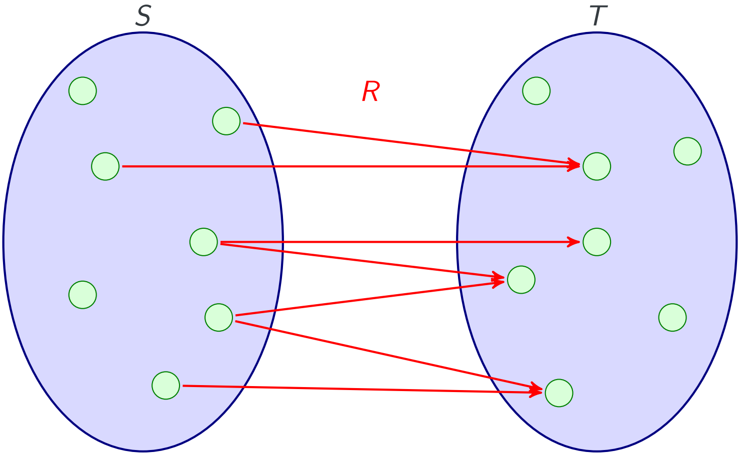
- **Composition**: If $R_1 \subseteq S \times T$ and $R_2 \subseteq T \times U$ then $R_1; R_2 \subseteq S \times U$:

$$R_1; R_2 \stackrel{\text{def}}{=} \{(s, u) \in S \times U \ : \ \text{there exists } t \in T \text{ such that } (s, t) \in R_1 \text{ and } (t, u) \in R_2\}.$$

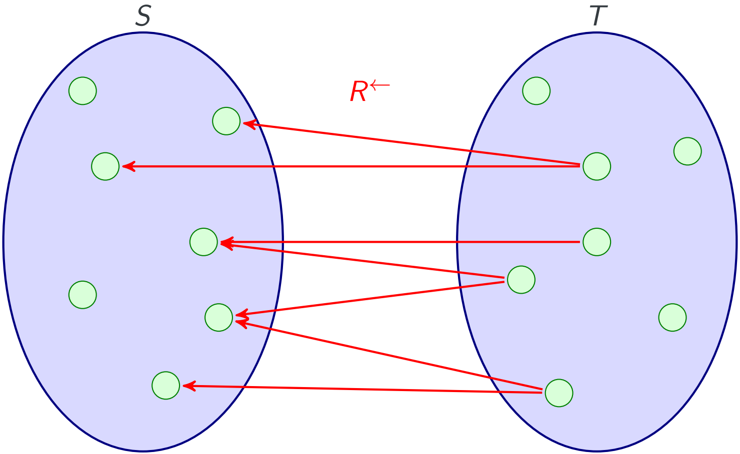
Fact

$(R^{\leftarrow})^{\leftarrow} = R$

Binary relation: Graphical representation



Binary relation: Graphical representation



Relational images

Given $R \subseteq S \times T$, $A \subseteq S$, and $B \subseteq T$.

Definition

- Relational image of A, $R(A)$:

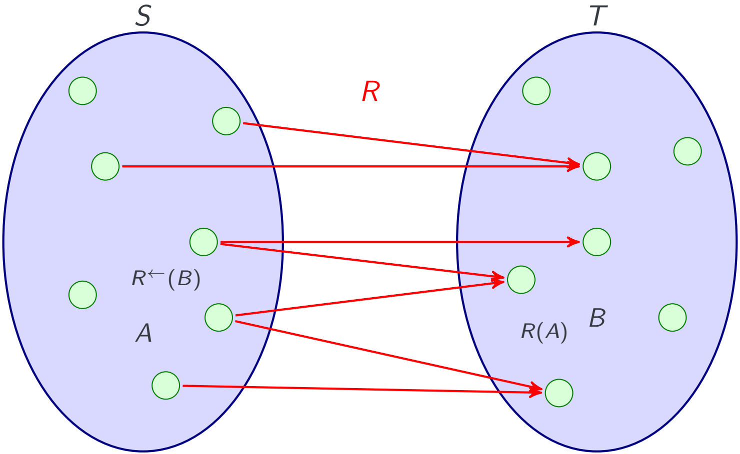
$$R(A) \stackrel{\text{def}}{=} \{t \in T \ : \ (s, t) \in R \text{ for some } s \in A\}$$

- Relational pre-image of B, $R^{\leftarrow}(B)$:

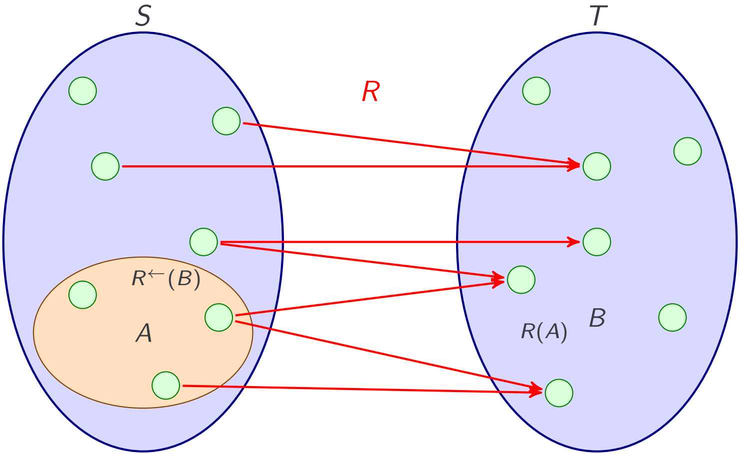
$$R^{\leftarrow}(B) \stackrel{\text{def}}{=} \{s \in S \ : \ (s, t) \in R \text{ for some } t \in B\}$$

Observe that the relational pre-image is the relational image of the converse relation.

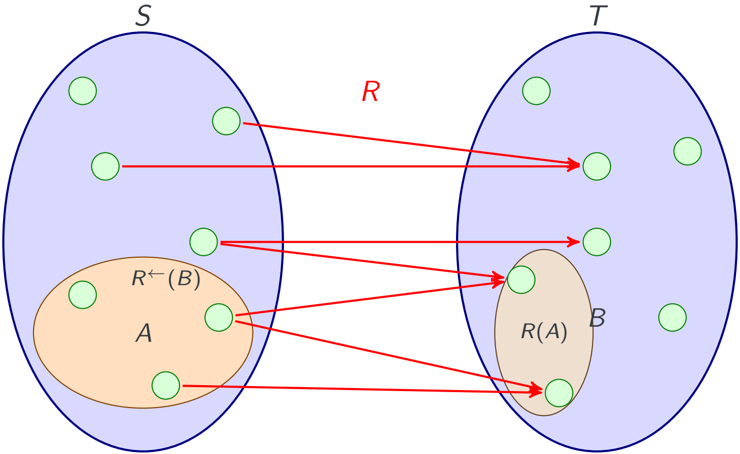
Binary relation: Graphical representation



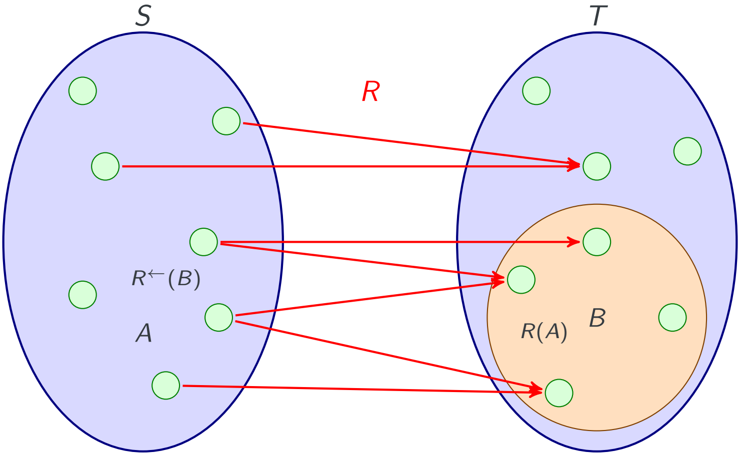
Binary relation: Graphical representation



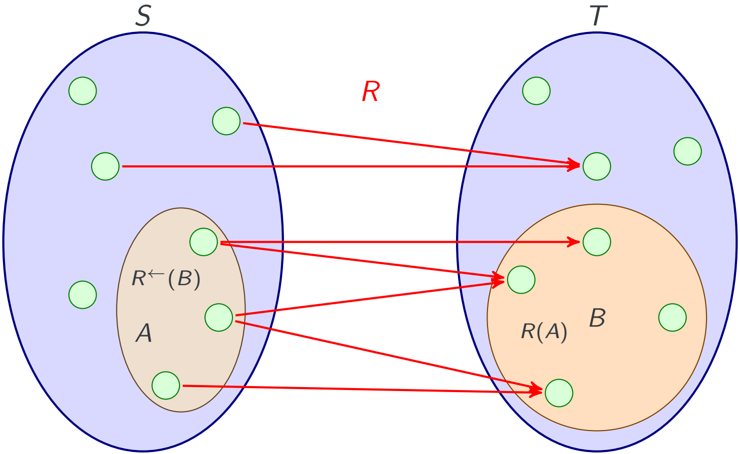
Binary relation: Graphical representation



Binary relation: Graphical representation



Binary relation: Graphical representation



Exercises

Exercises

Let $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{3, 4\}$, $X = [1, 4]$,
$M = \{A, B, C\}$, $N = \{A, B, C, X\}$.

- $|$ on X:
- $\in$ on $X \times M$:
- $\subseteq^{\leftarrow}$ on N:
- $|; \in$:
- $< (\{2\})$ (on X):

Exercises

Exercises

Let $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{3, 4\}$, $X = [1, 4]$,
$M = \{A, B, C\}$, $N = \{A, B, C, X\}$.

- $|$ on X: $\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4))\}$
- $\in$ on $X \times M$: $\{(1, A), (2, A), (2, B), (3, B), (3, C), (4, C)\}$
- $\subseteq^{\leftarrow}$ on N: $\{(A, A), (X, A), (B, B), (X, B), (C, C), (X, C), (X, X)\}$
- $|;\in$:
  $\{(1, A), (1, B), (1, C), (2, A), (2, B), (2, C), (3, B), (3, C), (4, C)\}$
- $<(\{2\})$ (on X): $\{3, 4\}$

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Properties of Binary Relations $R \subseteq S \times T$

A binary relation $R \subseteq S \times T$ is:

Definition

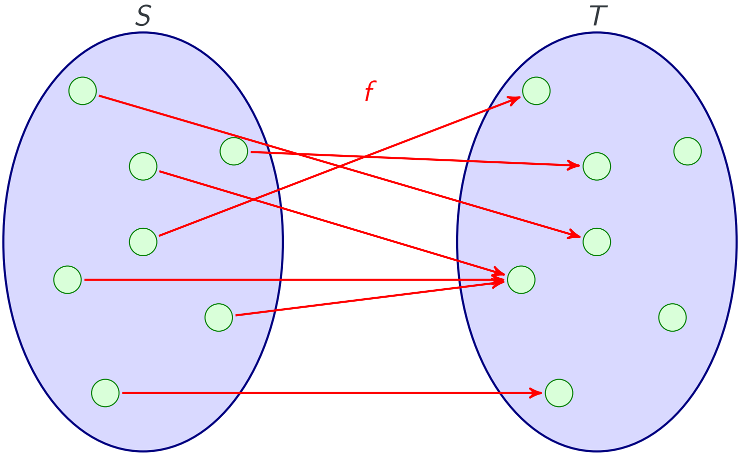
(Fun)	functional	For all $s \in S$ there is at most one $t \in T$ such that $(s, t) \in R$
(Tot)	total	For all $s \in S$ there is at least one $t \in T$ such that $(s, t) \in R$
(Inj)	injective	For all $t \in T$ there is at most one $s \in S$ such that $(s, t) \in R$
(Sur)	surjective	For all $t \in T$ there is at least one $s \in S$ such that $(s, t) \in R$
(Bij)	bijective	Injective and surjective

Functions and function properties

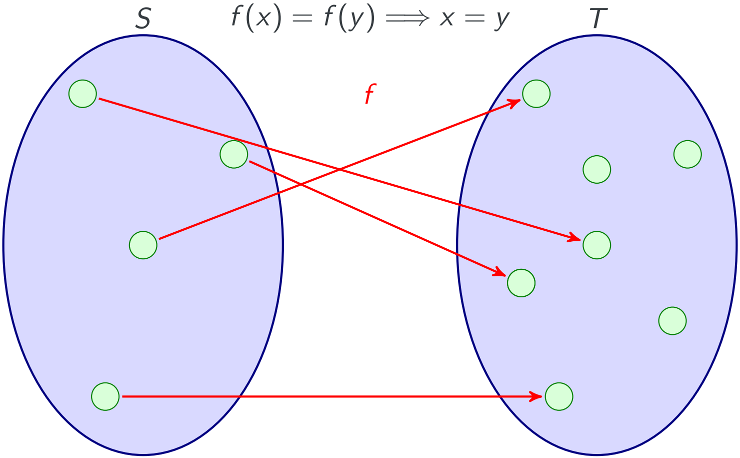
Definition

- **partial function** is a binary relation that is (Fun).
- A **function** is a binary relation that is (Fun) and (Tot).
- An **injection** is a function that is (Inj).
- A **surjection** is a function that is (Sur).
- A **bijection** is a function that is (Bij).

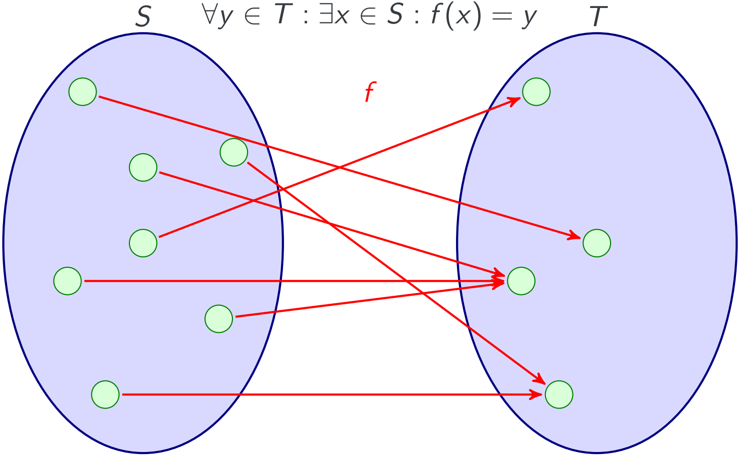
Graphical representation: Function



Graphical representation: Injection



Graphical representation: Surjection



Properties of Binary Relations $R \subseteq S \times S$

Definition

(R)	reflexive	For all $x \in S$: $(x, x) \in R$
(AR)	antireflexive	For all $x \in S$: $(x, x) \notin R$
(S)	symmetric	For all $x, y \in S$: If $(x, y) \in R$ then $(y, x) \in R$
(AS)	antisymmetric	For all $x, y \in S$: If (x, y) and $(y, x) \in R$ then $x = y$
(T)	transitive	For all $x, y, z \in S$: If (x, y) and $(y, z) \in R$ then $(x, z) \in R$

NB

- *Properties have to hold for all elements*
- *(S), (AS), (T) are conditional statements – they will hold if there is nothing which satisfies the 'if' part*

Relation properties: Examples

Examples

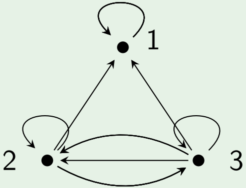
(R) Reflexivity: $(x, x) \in R$ for all x

(AR) Antireflexivity: $(x, x) \notin R$ for all x

(S) Symmetry: If $(x, y) \in R$ then $(y, x) \in R$ for all x, y

(AS) Antisymmetry: $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$ for all x, y

(T) Transitivity: $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$ for all x, y, z.



$$\begin{bmatrix} \bullet\circ & \circ & \circ \\ \circ\bullet & \bullet\circ & \circ\bullet \\ \bullet\circ & \circ\bullet & \bullet\circ \end{bmatrix}$$

Interaction of Properties

A relation *can* be both symmetric and antisymmetric. Namely, when R consists only of some pairs $(x, x), x \in S$.
A relation *cannot* be simultaneously reflexive and antireflexive (unless $S = \emptyset$).

NB

$$\left.\begin{array}{r}\textit{nonreflexive}\\ \textit{nonsymmetric}\end{array}\right\}\ \textit{is not the same as}\ \left\{\begin{array}{l}\textit{antireflexive/irreflexive}\\ \textit{antisymmetric}\end{array}\right.$$

Exercises

Exercises

$\boxed{\text{RW: 3.1.1}}$ The following relations are on $S = \{1, 2, 3\}$.
Which of the properties (R), (AR), (S), (AS), (T) does each satisfy?

(a) $(m, n) \in R$ if $m + n = 3$?

(e) $(m, n) \in R$ if $\max\{m, n\} = 3$?

Exercises

Exercises

$\boxed{\text{RW: 3.1.1}}$ The following relations are on $S = \{1, 2, 3\}$.
Which of the properties (R), (AR), (S), (AS), (T) does each satisfy?

(a) $(m, n) \in R$ if $m + n = 3$? (AR) and (S)

(e) $(m, n) \in R$ if $\max\{m, n\} = 3$? (S)

Exercises

$\boxed{\text{RW: 3.1.10}}$ Give examples of relations with specified properties.

(a) (AS), (T), not (R)

(b) (S), not (R), not (T)

Exercises

Exercises

$\boxed{\text{RW: 3.1.10}}$ Give examples of relations with specified properties.

(a) (AS), (T), not (R)

- Strict order of numbers $x < y$
- $\leq$ but with some pairs (x, x) removed
- Being a prime divisor: $(p, n) \in R$ iff p is prime and $p|n$
  - Not reflexive: $(1, 1) \notin R$
  - Transitivity is meaningful only for the pairs $(p, p), (p, n)$ $p|n$ for p prime

(b) (S), not (R), not (T)

Simplest example - inequality

Exercises

Exercises

RW: 3.6.10 (supp)

R is a relation on $\mathbb{N} \times \mathbb{N}$, i.e. it is a subset of $\mathbb{N}^2 \times \mathbb{N}^2$
$(m, n)\, R\, (p, q)$ if $m =_{(3)} p$ or $n =_{(5)} q$.

(a) Is R reflexive?

(b) Is R symmetric?

(c) Is R transitive?

Exercises

Exercises

RW: 3.6.10 (supp)

R is a relation on $\mathbb{N} \times \mathbb{N}$, i.e. it is a subset of $\mathbb{N}^2 \times \mathbb{N}^2$
$(m, n)\, R\, (p, q)$ if $m =_{(3)} p$ or $n =_{(5)} q$.

(a) Is R reflexive? Yes: $m =_{(3)} m$ so $(m, n)R(m, n)$.

(b) Is R symmetric? Yes: by symmetry of $\cdot =_{(n)} \cdot$.

(c) Is R transitive? No: Consider $(1, 1)$, $(1, 4)$ and $(2, 4)$.

Exercises

Exercises

Complete the following table of common relations (over $\mathbb{Z}$) and their properties:

	(R)	(AR)	(S)	(AS)	(T)
$=$					
$\leq$					
$<$					
$\emptyset$					
$\mathcal{U} = \mathbb{Z} \times \mathbb{Z}$					
$\mid$					
$=_{(3)}$					

Exercises

Exercises

Complete the following table of common relations (over $\mathbb{Z}$) and their properties:

	(R)	(AR)	(S)	(AS)	(T)	
$=$	$\checkmark$		$\checkmark$	$\checkmark$	$\checkmark$	
$\leq$	$\checkmark$			$\checkmark$	$\checkmark$	
$<$		$\checkmark$		$\checkmark$	$\checkmark$	
$\emptyset$		$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	
$\mathcal{U} = \mathbb{Z} \times \mathbb{Z}$	$\checkmark$		$\checkmark$		$\checkmark$	
$\mid$	$\checkmark$			$*$	$\checkmark$	$*$ True for $\mid \subseteq \mathbb{N} \times \mathbb{N}$
$=_{(3)}$	$\checkmark$		$\checkmark$		$\checkmark$	

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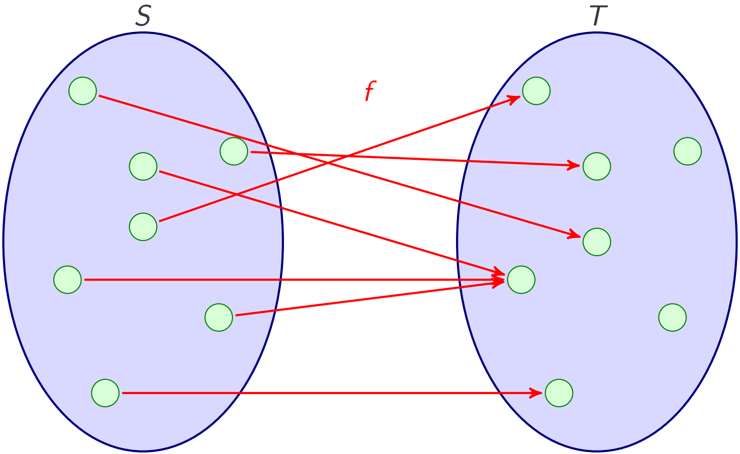
Definition

A **function**, $f : S \to T$, is a binary relation $f \subseteq S \times T$ that satisfies (Fun) and (Tot). That is, for all $s \in S$ there is *exactly one* $t \in T$ such that $(s, t) \in f$.

We write $f(s)$ for the unique element related to s.

We write T^S for the set of all functions from S to T.

Graphical representation



Functions

$f : S \longrightarrow T$ describes pairing of the sets: it means that f assigns to every element $s \in S$ a unique element $t \in T$. To emphasise where a specific element is sent, we can write $f : x \mapsto y$, which means the same as $f(x) = y$

		Symbol	
S	**domain** of f	$\mathrm{Dom}(f)$	(inputs)
T	**co-domain** of f	$\mathrm{Codom}(f)$	(*possible* outputs)
$f(S)$	**image** of f	$\mathrm{Im}(f)$	(*actual* outputs)
$= \{ f(x) : x \in \mathrm{Dom}(f) \}$			

Important!

The domain and co-domain are critical aspects of a function's definition.

$$f : \mathbb{N} \to \mathbb{Z} \quad \text{given by} \quad f(x) \mapsto x^2$$

and

$$g : \mathbb{N} \to \mathbb{N} \quad \text{given by} \quad g(x) \mapsto x^2$$

are different functions even though they have the same behaviour!

Converse of a function

Question

$f^{\leftarrow}$ *is a relation; when is it a function?*

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Weekly Feedback

I would appreciate any comments/suggestions/requests you have on this week's lectures.

https://forms.office.com/r/H0QWe5i5TG