Bayes decisions

1. Decisions under risk

2. Bayes decisions
Decision problem classes

Decision problems can be classified based on an agent’s epistemic state:

- **Decisions under certainty**: the agent knows the actual state
- **Decisions under uncertainty**:  
  - Decisions under *ignorance* (full uncertainty): the agent believes multiple states/outcomes are possible; likelihoods unknown  
  - Decisions under *risk*: the agent believes multiple states/outcomes are possible; likelihood information available
Example (River logistics)
Alice’s warehouse is located at X on a river that flows down-stream from C to A. She delivers goods to a client at C via motor boats. On some days a (free) goods ferry travels up the river, stopping at A then B and C, but not at X.

The fuel required (litres) to reach C from each starting point:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>X</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>To C</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Alice wants to minimise fuel consumption (in litres).

Example (Ferry likelihood)
Suppose Alice has received an order for a package to be delivered to C every day for the next eight days. Her records show that out of the last 100 days, the ferry was operating on 75.

- Additional information (Alice’s records) can be used to estimate likelihood of ferry being operational on any given day
- Single decision spans multiple ‘trials’ (days)
- How might this affect Alice’s decision?
River example

Alice considers three possible ways to get to C (from starting point X):

A : via A, by floating down the river
B : via B, by travelling up-stream to B
C : by travelling all the way to C

Outcomes are measured in litres left in a four-litre tank.

Exercise

Let \( w : \Omega \rightarrow \mathbb{R} \) denote fuel consumption in litres. What transformation \( f : \mathbb{R} \rightarrow \mathbb{R} \) is responsible for the values \( v : \Omega \rightarrow \mathbb{R} \) in the decision table?

Single decision; multiple trials

- Fuel savings when ferry operates on six of the eight days:
  
<table>
<thead>
<tr>
<th></th>
<th>( f )</th>
<th>( \bar{f} )</th>
<th>( \sum )</th>
<th>Avg</th>
<th>min</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>24</td>
<td>0</td>
<td>24</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>18</td>
<td>2</td>
<td>20</td>
<td>2.5</td>
<td>2</td>
</tr>
</tbody>
</table>

- Can we assume the ferry will operate in six of the eight days?
- Total fuel saved:
  A: \( 24 = 6 \times 4 + 2 \times 0 \)
  B: \( 20 = 6 \times 3 + 2 \times 1 \)
- Maximin choice based on least favourable state (\( \bar{f} \))
- Given information about likelihood of \( f \), is Maximin suitable?
Single decision; multiple trials

Simplifying assumptions:
- In how many of the next eight days will ferry operate: six? five? eight? none?
- Assume long sequence of days and maximum likelihood probability (six out of eight)
- Infer probability that ferry operates on any given day: \( p = \frac{75}{100} = \frac{3}{4} \)

\[
\begin{array}{c|ccc}
 & f & \bar{f} & E \\
A & 4 & 0 & 3 \\
B & 3 & 1 & 2.5 \\
\end{array}
\]

Likelihood and decisions

Alice’s total/average value is greater via A than B

Summary:
- In this situation there are multiple trials (days) of some random process (ferry operation)
- Different states may occur in each trial (day): ferry \((f)\) or no ferry \((\bar{f})\)
- Information available about ‘likelihood’ of occurrence of states: 75% ferry to 25% no ferry
- Maximin assumes worst case for each action even when the worst case (no ferry) is unlikely; \(i.e.,\) it ignores likelihoods
- Would like a decision rule which takes likelihood information into account
Probabilistic lotteries

**Definition (Probabilistic lottery)**

A *probabilistic lottery* over a finite set of outcomes, or *prizes*, $\Omega$, is a pair $\mathcal{L} = (\Omega, P)$, where $P : \Omega \rightarrow \mathbb{R}$ is a probability function. The lottery $\mathcal{L}$ is written:

$$\mathcal{L} = [p_1 : c_1 | p_2 : c_2 | \ldots | p_n : c_n]$$

where for each $s_i \in S \subseteq \mathcal{P}(\Omega)$, $p_i = P(s_i) = P(c_i)$.

**Example (To C via A)**

Alice’s decision to travel via A corresponds to:

$$\mathcal{L}_A = [\frac{3}{4} : 4 | \frac{1}{4} : 0]$$

where outcomes have been replaced by their values.

Value of a lottery

**Definition (Value of a lottery)**

The value of a probabilistic lottery $(\Omega, P, v)$ is the expected value over its outcomes:

$$V_v(\mathcal{L}) = E(v) = \sum_{\omega \in \Omega} P(\omega)v(\omega)$$

- For strategy A:

  $$V(\mathcal{L}_A) = \frac{3}{4}(4) + \frac{1}{4}(0) = 3 + 0 = 3$$

- Note: not value of any outcome of strategy A: 4, 0
- Frequency interpretation: $V(\mathcal{L}_A)$ is the average value of A over many days
Under risk, each strategy in a decision problem corresponds to a probabilistic lottery.

Definition (Bayes value)
Given a probability distribution over states, the *Bayes value*, $V_B$, of a strategy is the expected value of its outcomes.

Definition (Bayes strategy)
A *Bayes strategy* is a strategy with maximal *Bayes value*.

Definition (Bayes decision rule)
The *Bayes decision rule* is the rule which selects all the *Bayes* strategies.
Bayes strategies: River problem

Assume probability of ferry operating on an arbitrary day is \( p_f = p \):

\[
\begin{array}{c|cc|c}
& f & \bar{f} & V_B \\
A & 4 & 0 & 4p \\
B & 3 & 1 & 2p + 1 \\
\end{array}
\]

Bayes values for each strategy plotted for all values of \( p \in [0, 1] \).

Exercise

For what values of \( p \) will the Bayes decision rule prefer A to B?

Indifference curves: Maximin

For the pure actions below:

\[
\begin{array}{c|cc}
s_1 & s_2 \\
A & 2 & 3 \\
B & 4 & 0 \\
C & 3 & 3 \\
D & 5 & 2 \\
E & 3 & 5 \\
\end{array}
\]

Consider curves of all points representing strategies with same \textit{Maximin} value; \textit{i.e.}, \textit{Maximin indifference curves}. 
Indifference curves: Bayes

What do Bayes indifference curves look like?

<table>
<thead>
<tr>
<th></th>
<th>p</th>
<th>1 − p</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>f</td>
<td>f</td>
</tr>
<tr>
<td>B</td>
<td>a</td>
<td>v1  v2</td>
</tr>
</tbody>
</table>

\[ V_B = \begin{cases} 4p \\ 2p + 1 \end{cases} \]

Indifference curves:
\[ V_B(a) = pv_1 + (1 − p)v_2 = u \]

- In gradient-intercept form, \( v_2 = \frac{u}{1−p} − \frac{p}{1−p}v_1 \), where \( m = −\frac{p}{1−p} \); e.g., for \( p = \frac{3}{4} \), \( m = −\frac{3/4}{1/4} = −3 \)
- Because \( v_2 \propto u \); i.e., ‘higher’ lines receive greater Bayes values

In general, for two actions:

\[
p = \frac{\Delta y}{\Delta x + \Delta y} = \frac{m}{m − 1}
\]

where \( m \) is the gradient of line AB.

For example: if A is \((1, 3)\) and B is \((2, 1)\) then:

\[
p = \frac{3−1}{(2−1)+(3−1)} = \frac{2}{1+2} = \frac{2}{3}
\]
Bayes decisions

Indifference classes and *Bayes* decisions

**Exercises**

- Prove the expression for $p$
- For the river problem, what is the slope of the line joining the two actions?
- For what probability are the two actions of equal *Bayes* value?
- What is the *Bayes* value associated with this line?
- Repeat the above exercises for regret

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**Bayes strategies**

For the pure actions below with $P(s_1) = p$:

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$V_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>3</td>
<td>$3 - p$</td>
</tr>
<tr>
<td>B</td>
<td>5</td>
<td>1</td>
<td>$1 + 4p$</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>5</td>
<td>$5 - 2p$</td>
</tr>
</tbody>
</table>

Slope of BC: $m = \frac{5 - 1}{3 - 5} = -2$.

∴ $p = \frac{2}{2+1} = \frac{2}{3}$.

Note: $p \propto -m$. 
Bayes strategies: Probability plots

For the pure actions below with \( P(s_1) = p \):

<table>
<thead>
<tr>
<th></th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( V_B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>3</td>
<td>( 3 - p )</td>
</tr>
<tr>
<td>B</td>
<td>5</td>
<td>1</td>
<td>( 1 + 4p )</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>5</td>
<td>( 5 - 2p )</td>
</tr>
</tbody>
</table>

For \( p = \frac{2}{3} \), the value of the Bayes action(s) is least.

**Definition**

The *least favourable probability distribution* on the states/outcomes is the probability distribution for which Bayes strategies have minimal values.

Bayes solutions

For the pure actions below with \( P(s_1) = p \):

<table>
<thead>
<tr>
<th></th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( V_B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>5</td>
<td>( 5 - 4p )</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>1</td>
<td>( 1 + 3p )</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>4</td>
<td>( 4 - p )</td>
</tr>
</tbody>
</table>

Slope of BC: \( m = \frac{4 - 1}{3 - 4} = -3 \).

\[ \therefore p = \frac{3}{4} \]

Slope of AC: \( m = -\frac{1}{2} \).

\[ \therefore p = \frac{1}{3} \]
The Maximin action is a Bayes action when \( p = \frac{3}{4} \).

Mixed strategy \( a \sim 0.5A0.3B0.2C \) is not Bayes.

**Theorem**

Results about Bayes decision rule:

- Mixing can improve upon the Maximin value of pure strategies, but it does not improve upon the Bayes value of pure strategies.
- Bayes strategies are invariant/preserved under regret; i.e., the same strategy is chosen under regret as otherwise.

**Exercise**

Prove the theorems above.
## Bayes, Maximin, and admissibility

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>B</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

![Graphical representation]

### Exercises
- Which mixed strategies above are admissible?
- Are Maximin mixed strategies always admissible?
- Are Bayes mixed strategies always admissible?
- Are Maximin mixed strategies always Bayes for some $p$?
- Are admissible mixed strategies Bayes for some $p$?

## Bayes summary

- Partial (likelihood) information situations (*risk*)
- Information can affect degree of likelihood/belief (Bayesian probability)
- Bayes rule more appropriate when partial information available
- Bayes values, Bayes strategies, Bayes decision rule
- Graphical (visual) representation of Bayes strategies/values
- Bayes indifference curves