1. A *Boolean formula in Conjunctive Normal Form (CNF)* is a conjunction (AND) of disjunctions (OR) of literals (a Boolean variable or its negation).
   
   A *HORN* formula is a CNF formula where each clause contains at most one positive literal.

   For a CNF formula $F$ and an assignment $\tau : S \rightarrow \{0, 1\}$ to a subset $S$ of its variables, the formula $F[\tau]$ is obtained from $F$ by removing each clause that contains a literal that evaluates to 1 under $S$, and removing all literals that evaluate to 0 from the remaining clauses.

   **HORN-BACKDOOR DETECTION**
   
   Input: A CNF formula $F$ and an integer $k$.
   
   Parameter: $k$
   
   Question: Is there a subset $S$ of the variables of $F$ with $|S| \leq k$ such that for each assignment $\tau : S \rightarrow \{0, 1\}$, the formula $F[\tau]$ is a HORN formula?

   Example: $\lnot a \lor b \lor c \land (b \lor \lnot c \lor \lnot d) \land (a \lor b \lor \lnot e) \land (\lnot b \lor c \lor \lnot e)$ with $k = 1$ is a Yes-instance, certified by $S = \{b\}$.

   • Show that HORN-BACKDOOR DETECTION is FPT using the fact that Vertex Cover is FPT.

   **Hints**

   • Show the following: if two distinct positive literals occur in a same clause, then a HORN-backdoor must contain at least one of the corresponding variables.

   • Construct a parameterized reduction to Vertex Cover based on these pairwise conflicts.

2. Show that Weighted Circuit Satisfiability $\in \text{XP}$.

   **Hint**

   • There are $n^k$ assignments of weight $k$, where $n$ is the number of input gates.

3. Recall that a *$k$-coloring* of a graph $G = (V, E)$ is a function $f : V \rightarrow \{1, 2, \ldots, k\}$ assigning colors to $V$ such that no two adjacent vertices receive the same color.

   **Multicolor Clique**

   Input: A graph $G = (V, E)$, an integer $k$, and a $k$-coloring of $G$

   Parameter: $k$

   Question: Does $G$ have a clique of size $k$?

   • Show that Multicolor Clique is W[1]-hard.
Solution
The proof is by a parameterized reduction from CLIQUE.

Construction. Let \((G = (V, E), k)\) be an instance for CLIQUE. We construct an instance \((G' = (V', E'), k', f)\) for MULTICOLOR CLIQUE as follows. For each \(v \in V\), create \(k\) vertices \(v(1), \ldots, v(k)\) and add them to \(V'\). For every pair \(u(i), v(j) \in V'\) with \(i \neq j\), add \(uv(i)v(j)\) to \(E'\) if and only if \(uv \in E\).

Set \(k' := k\). Set \(f(v(i)) = i\) for each \(v \in V\) and \(i \in \{1, \ldots, k\}\).

Equivalence. \(G\) has a clique of size \(k\) if and only if \(G'\) has a clique of size \(k\).

\((\Rightarrow): \) Let \(S = \{s_1, \ldots, s_k\}\) be a clique in \(G\). Then \(S' = \{s_1(1), s_2(2), \ldots, s_k(k)\}\) is a clique in \(G'\) since \(s_is_j \in E\) implies \(s_i(s_j) \in E'\) in our construction.

\((\Leftarrow): \) Let \(S'\) be a clique of size \(k\) in \(G'\). Since for each \(i \in \{1, \ldots, k\}\), \(\{v_i : v \in V\}\) is an independent set in \(G'\), \(S'\) contains exactly one vertex from each color class of \(f\). Denote \(S' = \{s'_1(1), \ldots, s'_k(k)\}\). Then, \(S = \{s_1, \ldots, s_k\}\) is a clique in \(G\).

Parameter. \(k' \leq k\).

Running time. The construction can clearly be done in FPT time, and even in polynomial time.

4. A set system \(S\) is a pair \((V, H)\), where \(V\) is a finite set of elements and \(H\) is a set of subsets of \(V\). A set cover of a set system \(S = (V, H)\) is a subset \(X\) of \(H\) such that each element of \(V\) is contained in at least one of the sets in \(X\), i.e., \(\bigcup_{Y \in X} Y = V\).

### Set Cover

<table>
<thead>
<tr>
<th>Input:</th>
<th>A set system (S = (V, H)) and an integer (k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter:</td>
<td>(k)</td>
</tr>
<tr>
<td>Question:</td>
<td>Does (S) have a set cover of cardinality at most (k)?</td>
</tr>
</tbody>
</table>

- Show that Set Cover is W[2]-hard.

**Hints** Reduce from Dominating Set:
- add an element for each vertex and
- add a set for each vertex, containing all the vertices in its closed neighborhood.

5. A hitting set of a set system \(S = (V, H)\) is a subset \(X\) of \(V\) such that \(X\) contains at least one element of each set in \(H\), i.e., \(X \cap Y \neq \emptyset\) for each \(Y \in H\).

### Hitting Set

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>Parameter:</td>
<td>(k)</td>
</tr>
<tr>
<td>Question:</td>
<td>Does (S) have a hitting set of size at most (k)?</td>
</tr>
</tbody>
</table>
• Show that Hitting Set is W[2]-hard.

**Solution sketch**
Reduce from Set Cover.
Let \((S = (V, H), k)\) be an instance for Set Cover.
Construct an instance \((S' = (V', H'), k)\) for Hitting Set:

- \(V' := H\)
- \(H' := \{\{h \in H : v \in h\} : v \in V\}\)