# COMP4418: Knowledge Representation and Reasoning

First-Order Logic 2

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# First-Order Logic

Before building system

before there can be learning, reasoning, planning, explanation ...

need to be able to express knowledge

Want a precise declarative language

- declarative: believe P = hold P to be **true**
  cannot believe P without some sense of what it would mean for the world to satisfy P
- precise: need to know exactly
  - what strings of symbols count as sentences
  - what it means for a sentence to be true
    (but without having to specify which ones are true)

What does it mean to have a language?

- syntax
- **semantics**
- pragmatics

Here: language of first-order logic

again: not the only choice



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# Semantics

How to interpret sentences?

- what do sentences claim about the world?
- what does believing one amount to?

Without answers, cannot use sentences to represent knowledge

Problem:

*cannot fully specify interpretation of sentences because non-logical symbols reach outside the language*

So:

*make clear dependence of interpretation on non-logical symbols*

Logical interpretation:

*specification of how to understand predicate and function symbols*

Can be complex!

*IsABetterJudgeOfCharacterThan,*
*favouriteIceCreamFlavourOf,*
*puddleOfWater27*

# Simple Case

There are objects

some satisfy predicate $P$; some do not

Each interpretation settles extension of $P$

borderline cases ruled in separate interpretations

Each interpretation assigns to function $f$ a mapping from objects to objects

functions always well-defined and single-valued

Main assumption:

- this is all you need to know about the non-logical symbols to understand which sentences of FOL are true or false
- In other words, given a specification of
  - what objects there are
  - which of them satisfy $P$
  - what mapping is denoted by $f$
- it will be possible to say which sentences of FOL are true and which are not

# Interpretations

Two parts: $I = \langle D, \Phi \rangle$

$D$ is the domain of discourse

- can be any set
- not just formal / mathematical objects
- e.g. people, tables, numbers, sentences, chunks of peanut butter, situations, the universe

$\Phi$ is an *interpretation* mapping

- If $P$ is a predicate symbol of arity $n$, $\Phi(P) \subseteq [D \times D \times \ldots \times D]$
  an $n$-ary relation over $D$
  Can view interpretation of predicates in terms of characteristic function
  $\Phi(P) \in [D \times D \times \ldots \times D \rightarrow \{0, 1\}]$
- If $f$ is a function symbol of arity $n$,
  $\Phi(f) \in [D \times D \times \ldots \times D \rightarrow D]$
  an $n$-ary function over $D$
- For constants, $\Phi(c) \in D$


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# Denotation

In terms of interpretation $I$, terms will denote elements of $D$.

will write element as $I$

For terms with variables, denotation depends on the values of variables

will write as $I, \mu||t||$

where $\mu \in [Variables \rightarrow D]$ called a variable assignment

Rules of interpretation:

1. $I, \mu||v|| = \mu(v)$
2. $I, \mu||f(t_1, t_2, \ldots, t_n)|| = H(d_1, d_2, \ldots, d_n)$

   *where* $H = \Phi(f)$
   *and* $d_i$ *recursively*

# Satisfaction

In terms of $I$, wffs will be true for some values of the free variables and false for others

*will write as $I, \mu \models \alpha$ "$\alpha$ is satisfied by $I$ and $\mu$" where $\mu \in [Variables \rightarrow D]$, as before*
*or $I \models \alpha$, when $\alpha$ is a sentence*
*or $I \models S$, when $S$ s a set of sentences (all sentences in $S$ are true in $I$).*

Rules of interpretation:

1. $I, \mu \models P(t_1, t_2, \ldots, t_n)$ iff $\langle d_1, d_2, \ldots, d_n \rangle$ where $R = \Phi(P)$ and $d_i = I, \mu||t_i||$, as on previous slide
2. $I, \mu \models (t_1 = t_2)$ iff $I, \mu||t_1||$ is the same as $I, \mu||t_2||$
3. $I, \mu \models \neg\alpha$ iff $I, \mu \not\models \alpha$
4. $I, \mu \models (\alpha \wedge \beta)$ iff $I, \mu \models \alpha$ and $I, \mu \models \beta$
5. $I, \mu \models (\alpha \vee \beta)$ iff $I, \mu \models \alpha$ or $I, \mu \models \beta$
6. $I, \mu \models \exists v, \alpha$ iff for some $d \in D$, $I, \mu\{d; v\} \models \alpha$
7. $I, \mu \models \forall v, \alpha$ iff for all $d \in D$, $I, \mu\{d; v\} \models \alpha$ where $\mu\{d; v\}$ is just like $\mu$, except on $v$, where $\mu(v) = d$.

For propositional subset: $I \models p$ iff $\Phi(p) = 1$ and the rest as above

# Logical Consequence

Semantic rules of interpretation tell us how to understand all wffs in terms of specification for non-logical symbols.

But some connections among sentences are independent of non-logical symbols involved.   e.g. If $\alpha$ is true under *I*, then so is $\neg(\beta \wedge \neg\alpha)$ no matter what *I* is, why $\alpha$ is true, what $\beta$ is,...   a function of logical symbols only

*S entails* $\alpha$ or $\alpha$ is a *logical consequence* of *S*:

$$S \models \alpha \text{ iff for every } I, \text{ if } I \models S, \text{ then } I \models \alpha$$

In other words: for no *I*, $I \models S \cup \{\neg\alpha\}$.

Say that $S \cup \{\neg\alpha\}$ is *unsatisfiable*

Special case: *S* is empty $\models \alpha$ iff for every *I*, $I \models \alpha$. Say $\alpha$ is *valid*.

Note: $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \models \alpha$ iff $\models (\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n) \rightarrow \alpha$

finite entailment reduces to validity

# Why do we care?

We do not have access to user-intended interpretation of non-logical symbols
But, with *entailment*, we know that if $S$ is true in the intended interpretation, then so is $\alpha$.

- If the user's view has the world satisfying $S$, then it must also satisfy $\alpha$
- There may be other sentences true also; but $\alpha$ is logically guaranteed.

So what about:

*Dog(fido)* $\Rightarrow$ *Mammal(fido)??*
*Not entailment!*
*There are logical interpretations where* $\Phi(Dog) \not\subset \Phi(Mammal)$
*Key idea of KR:*
*include such connections explicitly in S*
$\forall x[Dog(x) \rightarrow Mammal(x)]$
*Get:* $S \cup \{Dog(fido)\} \models Mammal(fido)$
*The rest is just the details...*

# Knowledge Bases

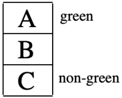
KB is set of sentences

explicit statement of sentences believed (including assumed connections among non-logical symbols)

KB $\models \alpha$

- $\alpha$ is a further consequence of what is believed
- explicit knowledge: KB
- implicit knowledge: $\{\alpha | \mathit{KB} \models \alpha\}$

Often non trivial: explicit $\rightarrow$ implicit




Example:

Three blocks stacked.
Top one is green.
Bottom one is not green.

Is there a green block directly on top of a non-green block?

# A Formalisation

$S = \{On(a,b), On(b,c), Green(a), \neg Green(c)\}$
all that is required
$\alpha = \exists x \exists y [Green(x) \wedge \neg Green(y) \wedge On(x,y)]$
**Claim:** $S \models \alpha$
**Proof:**
Let $I$ be any interpretation such that $I \models S$.
Case 1: $I \models Green(b)$.
$\therefore I \models Green(b) \wedge \neg Green(c) \wedge On(b,c)$.
$\therefore I \models \alpha$
Case 2: $I \not\models Green(b)$.
$\therefore I \models \neg Green(b)$
$\therefore I \models Green(a) \wedge \neg Green(b) \wedge On(a,b)$.
$\therefore I \models \alpha$
Either way, for any $I$, if $I \models S$ then $I \models \alpha$
So $S \models \alpha$. QED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# Knowledge-Based System

Start with (large) KB representing what is explicitly known
e.g. what the system has been told
Want to influence behaviour based on what is *implicit* in the KB (or as close as possible)
Requires reasoning

- *deductive inference:*
  process of calculating entailments of KB
  i.e given KB and any $\alpha$, determine if KB $\models \alpha$.
- Process is *sound* if whenever it produces $\alpha$ then KB $\models \alpha$
  does not allow for plausible assumptions that may be true in intended interpretation
- Process is *complete* if whenever KB $\models \alpha$, it produces $\alpha$
  does not allow for process to miss some $\alpha$ or be unable to determine the status of $\alpha$