4. Inclusion-Exclusion

COMP6741: Parameterized and Exact Computation

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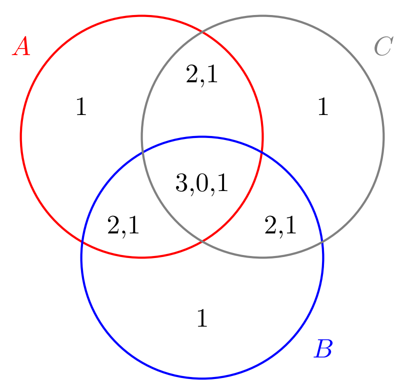
Contents

1 The Principle of Inclusion-Exclusion

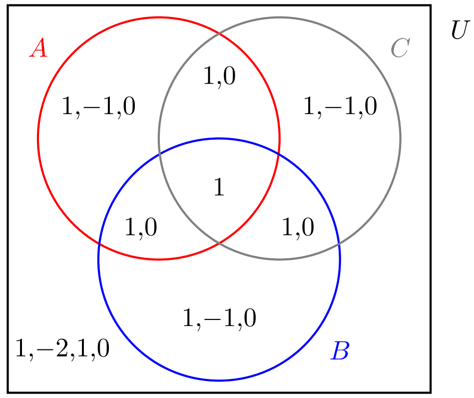
... for 3 sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$|A \cup B \cup C| = \sum_{X \subseteq \{A,B,C\}} (-1)^{|X|+1} \cdot \left|\bigcap X\right|$$



... intersection version

$$|A \cap B \cap C| = |U| - |\overline{A}| - |\overline{B}| - |\overline{C}| + |\overline{A} \cap \overline{B}| + |\overline{A} \cap \overline{C}| + |\overline{B} \cap \overline{C}| - |\overline{A} \cap \overline{B} \cap \overline{C}|$$
$$|A \cap B \cap C| = \sum_{X \subseteq \{A,B,C\}} (-1)^{|X|} \cdot \left|\bigcap \overline{X}\right|$$



Inclusion-Exclusion Principle – intersection version

Theorem 1 (IE-theorem – intersection version)**.** *Let $U = A_0$ be a finite set, and let $A_1, \ldots, A_k \subseteq U$.*

$$\left| \bigcap_{i \in \{1,\ldots,k\}} A_i \right| = \sum_{J \subseteq \{1,\ldots,k\}} (-1)^{|J|} \left| \bigcap_{i \in J} \overline{A_i} \right|,$$

where $\overline{A_i} = U \setminus A_i$ and $\bigcap_{i \in \emptyset} = U$.

Proof sketch.

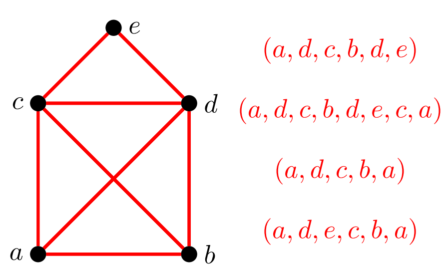
- An element $e \in \bigcap_{i \in \{1,\ldots,k\}} A_i$ is counted on the right only for $J = \emptyset$.
- An element $e \notin \bigcap_{i \in \{1,\ldots,k\}} A_i$ is counted on the right for all $J \subseteq I$, where I is the set of indices i such that $e \notin A_i$.
  - counted negatively for each odd-sized $J \subseteq I$, and positively for each even-sized $J \subseteq I$
  - a non-empty set has as many even-sized subsets as odd-sized subsets

□

2 Counting Hamiltonian Cycles

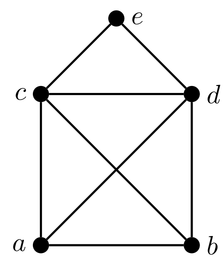
Walks and cycles

- A *walk* of length k in a graph $G = (V, E)$ (short, a *k-walk*) is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that $v_i v_{i+1} \in E$ for each $i \in \{0, \ldots, k-1\}$.
- A walk $(v_0, v_1, \ldots, v_k)$ is *closed* if $v_0 = v_k$.
- A *cycle* is a 2-regular subgraph of G.
- A *Hamiltonian cycle* of G is a cycle of length $n = |V|$.



#Hamiltonian-Cycles

#HAMILTONIAN-CYCLES	
Input:	A graph $G = (V, E)$
Output:	The number of Hamiltonian cycles of G



This graph has *2* Hamiltonian cycles.

IE for #Hamiltonian-Cycles

- U: the set of closed n-walks starting at vertex 1
- $A_v \subseteq U$: walks in U that visit vertex $v \in V$
- $\Rightarrow$ number of Hamiltonian cycles is $|\bigcap_{v \in V} A_v|$
- To use the IE-theorem, we need to compute $|\bigcap_{v \in S} \overline{A_v}|$, the number of walks from U in the graph $G - S$.

A simpler problem

#CLOSED n-WALKS	
Input:	An integer n, and a graph $G = (V, E)$ on $\leq n$ vertices
Output:	The number of closed n-walks in G starting at vertex 1

Dynamic programming

- $T[d, v]$: number of d-walks starting at vertex 1 and ending at vertex v
- Base cases: $T[0, 1] = 1$ and $T[0, v] = 0$ for all $v \in V \setminus \{1\}$
- DP recurrence: $T[d, v] = \sum_{uv \in E} T[d - 1, u]$
- Table T is filled by increasing d
- Return $T[n, 1]$ in $O(n^3)$ time

Wrapping up

- Recall:
  U: set of closed n-walks starting at vertex 1
  A_v: set of closed n-walks that start at vertex 1 and visit vertex v
- By the IE-theorem, the number of Hamiltonian cycles is

$$\left|\bigcap_{v \in V} A_v\right| = \sum_{S \subseteq V} (-1)^{|S|} \left|\bigcap_{v \in S} \overline{A_v}\right|$$

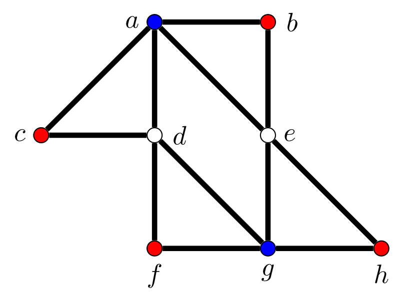
- We have seen that $\left|\bigcap_{v \in S} \overline{A_v}\right|$ can be computed in $O(n^3)$ time.
- So, $\sum_{S \subseteq V} (-1)^{|S|} \left|\bigcap_{v \in S} \overline{A_v}\right|$ can be evaluated in $O(2^n n^3)$ time

Theorem 2. #HAMILTONIAN-CYCLES *can be solved in* $O(2^n n^3)$ *time and polynomial space, where* $n = |V|$.

3 Coloring

A *k-coloring* of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, ..., k\}$ assigning colors to V such that no two adjacent vertices receive the same color.

COLORING	
Input:	Graph G, integer k
Question:	Does G have a k-coloring?



Exercise

- Suppose A is an algorithm solving COLORING in $O(f(n))$ time, $n = |V|$, where f is non-decreasing.
- Design a $O^*(f(n))$ time algorithm B, which, for an input graph G, finds a coloring of G with a minimum number of colors.

Solution (sketch)

1. First, compute the smallest number of colors needed to color G
   - For $k = 1$ to n, execute algorithm A for the instance (G, k), and stop when encountering the first YES-instance.
   - (Alternatively, use binary search to find the smallest k for which (G, k) is a YES-instance)
2. Now, compute an actual k-coloring using the following ideas
   - Select two non-adjacent vertices u and v, and check whether G as a k-coloring where u and v receive distinct colors.
     This can be done by adding an edge between u and v, and using algorithm A.
     If there is such a k-coloring, add the edge uv, and continue with two other distinct vertices.
     If not, then u and v must receive the same color, and we merge them into a single vertex, and continue by picking two new non-adjacent vertices
   - A complete graph on n vertices needs n colors.

IE formulation

Observation: partitioning vs. covering

$$G = (V, E) \text{ has a } k\text{-coloring} \Leftrightarrow G \text{ has independent sets } I_1, \ldots, I_k \text{ such that } \bigcup_{i=1}^{k} I_i = V.$$

- U: set of tuples $(I_1, \ldots, I_k)$, where each I_i, $i \in \{1, \ldots, k\}$, is an independent set
- $A_v = \{(I_1, \ldots, I_k) \in U : v \in \bigcup_{i \in \{1,\ldots,k\}} I_i\}$
- Note: $\left|\bigcap_{v \in V} A_v\right| \neq 0 \Leftrightarrow G$ has a k-coloring
- To use the IE-theorem, we need to compute

  $$\begin{aligned}\left|\bigcap_{v \in S} \overline{A_v}\right| &= |\{(I_1, \ldots, I_k) \in U : I_1, \ldots, I_k \subseteq V \setminus S\}| \\ &= s(V \setminus S)^k,\end{aligned}$$

  where $s(X)$ is the number of independent sets in $G[X]$

A simpler problem

#IS of Induced Subgraphs	
Input:	A graph $G = (V, E)$
Output:	$s(X)$, the number of independent sets of $G[X]$, for each $X \subseteq V$

Dynamic Programming

- $s(X)$: the number of independent sets of $G[X]$
- Base case: $s(\emptyset) = 1$
- DP recurrence: $s(X) = s(X \setminus N_G[v]) + s(X \setminus \{v\})$, where $v \in X$
- Table s filled by increasing cardinalities of X
- Output $s(X)$ for each $X \subseteq V$ in time $O^*(2^n)$

Wrapping up
Now, evaluate

$$\left|\bigcap_{v\in V} A_v\right| = \sum_{S\subseteq V} (-1)^{|S|} \left|\bigcap_{v\in S} \overline{A_v}\right| = \sum_{S\subseteq V} (-1)^{|S|} s(V \setminus S)^k,$$

in $O^*(2^n)$ time. G has a k-coloring iff $\left|\bigcap_{v\in V} A_v\right| > 0$.

Theorem 3 ([Bjørklund & Husfeldt '06], [Koivisto '06])**.** Coloring *can be solved in* $O^*(2^n)$ *time (and space).*

Corollary 4. *For a given graph* G*, a coloring with a minimum number of colors can be found in* $O^*(2^n)$ *time (and space).*

... polynomial space
Using an algorithm by [Wahlström 2008], counting all independent sets in a graph on n vertices in $O(1.2377^n)$ time, we obtain a polynomial-space algorithm for Coloring with running time

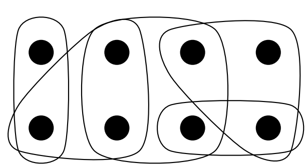
$$\sum_{S\subseteq V} O(1.2377^{n-|S|}) = \sum_{s=0}^{n} \binom{n}{s} O(1.2377^{n-s}) = O(2.2377^n).$$

Here, we used the Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

Theorem 5. Coloring *can be solved in* $O(2.23772^n)$ *time and polynomial space.*

4 Counting Set Covers

#Set Covers	
Input:	A finite ground set V of elements, a collection H of subsets of V, and an integer k
Output:	The number of ways to choose a k-tuple of sets $(S_1, \ldots, S_k)$ with $S_i \in H$, $i \in \{1, \ldots, k\}$, such that $\bigcup_{i=1}^{k} S_i = V$.



This instance has $1 \cdot 3! = 6$ covers with 3 sets and $3 \cdot 4! = 72$ covers with 4 sets.

We consider, more generally, that H is given only implicitly, but can be enumerated in $O^*(2^n)$ time and space.

Algorithm for Counting Set Covers

- U: set of k-tuples $(S_1, \dots, S_k)$, where $S_i \in H$, $i \in \{1, \dots, k\}$,
- $A_v = \{(S_1, \dots, S_k) \in U \, : \, v \in \bigcup_{i \in \{1,\dots,k\}} S_i\}$,
- the number of covers with k sets is

$$\left|\bigcap_{v \in V} A_v\right| = \sum_{S \subseteq V} (-1)^{|S|} \left|\bigcap_{v \in S} \overline{A_v}\right|$$
$$= \sum_{S \subseteq V} (-1)^{|S|} s(V \setminus S)^k,$$

where $s(X)$ is the number of sets in H that are subsets of X.

Compute $s(X)$
For each $X \subseteq V$, compute $s(X)$, the number of sets in H that are subsets of X.

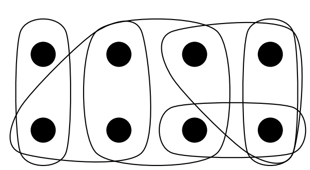
Dynamic Programming

- Arbitrarily order $V = \{v_1, v_2, \dots, v_n\}$
- $g[X, i] = |\{S \in H : (X \cap \{v_i, \dots, v_n\}) \subseteq S \subseteq X\}|$
- Note: $g[X, n+1] = s(X)$
- Base case: $g[X, 1] = \begin{cases} 1 & \text{if } X \in H \\ 0 & \text{otherwise.} \end{cases}$
- DP recurrence: $g[X, i] = \begin{cases} g[X, i-1] & \text{if } v_{i-1} \notin X \\ g[X \setminus \{v_{i-1}\}, i-1] + g[X, i-1] & \text{otherwise.} \end{cases}$
- Table filled by increasing i

Theorem 6. #SET COVERS *can be solved in* $O^*(2^n)$ *time and space, where* $n = |V|$.

5 Counting Set Partitions

#ORDERED SET PARTITIONS	
Input:	A finite ground set V of elements, a collection H of subsets of V, and an integer k
Output:	The number of ways to choose a k-tuple of *pairwise disjoint* sets $(S_1, \dots, S_k)$ with $S_i \in H$, $i \in \{1, \dots, k\}$, such that $\bigcup_{i=1}^{k} S_i = V$. *(Now,* $S_i \cap S_j = \emptyset$*, if* $i \neq j$*.)*



This instance has $1 \cdot 3! = 6$ ordered partitions with 3 sets.

IE formulation

Lemma 7. *The number of ordered k-partitions of a set system (V, H) is*

$$\sum_{S \subseteq V} (-1)^{|S|} a_k(V \setminus S),$$

where $a_k(X)$ denotes the number of k-tuples of sets $S_1, \dots, S_k \subseteq X$ with $\sum_{i=1}^{k} |S_i| = |V|$.

Proof (Sketch).

- U: set of tuples $(S_1, \dots, S_k)$, where $S_i \in H$, $i \in \{1, \dots, k\}$, *and* $\sum_{i=1}^{k} |S_i| = |V|$
- $A_v = \{(S_1, \dots, S_k) \in U : v \in \bigcup_{i \in \{1,\dots,k\}} S_i\}$,
- the number of ordered partitions with k sets is

$$\left| \bigcap_{v \in V} A_v \right| = \sum_{S \subseteq V} (-1)^{|S|} \left| \bigcap_{v \in S} \overline{A_v} \right| = \sum_{S \subseteq V} (-1)^{|S|} a_k(V \setminus S).$$

□

IE evaluation

For each $X \subseteq V$, we need to compute $a_k(X)$, the number of k-tuples of sets $S_1, \dots, S_k \subseteq X$ with $\sum_{i=1}^{k} |S_i| = |V|$.

Dynamic Programming

(1) Compute $s[X, i] = |\{Y \in H : Y \subseteq X \text{ and } |Y| = i\}|$ for each $X \subseteq V$ and each $i \in \{0, \dots, n\}$:

- The entries $s[\cdot, i]$ are computed the same ways as $s[\cdot]$ in the previous section, but keep only the sets in H of size i.

(2) $A[\ell, m, X]$: number of tuples $(S_1, \dots, S_\ell)$ with $S_i \in H$, $S_i \subseteq X$, and $\sum_{i=1}^{\ell} |S_i| = m$.

- Base case: $A[1, m, X] = s[X, m]$
- DP recurrence: $A[\ell, m, X] = \sum_{i=1}^{m-1} s[X, i] \cdot A[\ell - 1, m - i, X]$
- Table filled by increasing ℓ
- Note: $a_k(X) = A[k, |V|, X]$

Algorithm for Counting Set Partitions

Theorem 8. #ORDERED SET PARTITIONS *can be solved in $O^*(2^n)$ time and space.*

Corollary 9. *There is an algorithm computing the number of k-colorings of an input graph on n vertices in $O^*(2^n)$ time and space.*

Covering and partitioning in polynomial space

Theorem 10. *The number of covers with k sets and the number of ordered partitions with k sets of a set system (V, H) can be computed in polynomial space and*

1. *$O^*(2^n |H|)$ time, assuming that H can be enumerated in $O^*(|H|)$ time and polynomial space*
2. *$O^*(3^n)$ time, assuming membership in H can be decided in polynomial time, and*
3. *$\sum_{j=0}^{n} \binom{n}{j} T_H(j)$ time, assuming there is a $T_H(j)$ time and polynomial space algorithm to count for any $W \subseteq V$ with $|W| = j$ the number of sets $S \in H$ satisfying $S \cap W = \emptyset$.*

Exercise
A graph $G = (V, E)$ is *bipartite* if V can be partitioned into two independent sets. A *matching* in a graph $G = (V, E)$ is a set of edges $M \subseteq E$ such that no two edges of M have an end-point in common. The matching M in G is *perfect* if every vertex of G is contained in an edge of M.

#Bipartite Perfect Matchings	
Input:	Bipartite graph $G = (V, E)$
Output:	The number of perfect matchings in G.

1. Design an algorithm with running time $O^*\left(\left(\frac{n}{2}\right)!\right)$, where $n = |V|$.
2. Design a polynomial-space $O^*(2^{n/2})$-time inclusion-exclusion algorithm.

Solution (sketch)

1. Let (X, Y) be a bipartition of V such that X and Y are independent sets If $|X| \neq |Y|$, then return 0. Denote $X = \{x_1, \ldots, x_{n/2}\}$ and $Y = \{y_1, \ldots, y_{n/2}\}$. For each permutation $\pi = (y_{\pi(1)}, \ldots, y_{\pi(n/2)})$ of Y,
$$\{x_i y_{\pi(i)} : 1 \le i \le n/2\}$$
is a perfect matching iff $x_i y_{\pi(i)} \in E$ for each $i \in \{1, \ldots, n/2\}$.
2. U: contains each set of $n/2$ edges $\{e_1, \ldots, e_{n/2}\}$ such that $x_i \in e_i$. For $v \in Y$, $A_v = \{S \in U : v \in \bigcup S\}$. The number of perfect matchings is
$$\begin{aligned}\left|\bigcap_{v \in Y} A_v\right| &= \sum_{S \subseteq Y} (-1)^{|S|} \left|\bigcap_{v \in S} \overline{A_v}\right| \\ &= \sum_{S \subseteq Y} (-1)^{|S|} \prod_{i=1}^{n/2} |N(x_i) \setminus S|.\end{aligned}$$

6 Further Reading

- Chapter 4, *Inclusion-Exclusion* in Fedor V. Fomin and Dieter Kratsch. Exact Exponential Algorithms. Springer, 2010.
- Thore Husfeldt. Invitation to Algorithmic Uses of Inclusion-Exclusion. Proceedings of the 38th International Colloquium on Automata, Languages and Programming (ICALP 2011): 42-59, 2011.

Advanced Reading

- Chapter 7, *Subset Convolution* in Fedor V. Fomin and Dieter Kratsch. Exact Exponential Algorithms. Springer, 2010.