1. Hypotrochoids

A *hypotrochoid* is the curve obtained by tracing the positions taken by a point $P$ rigidly attached to a circle $C_2$ of centre $C_2$ and radius $r$, $P$ being at a distance $d$ from $C_2$, with $C_2$ rolling around the inside of another circle $C_1$ of centre $C_1$ and radius $R$. To compute the equation of the curve, one assumes that $C_1$ is located at the origin of the plane, so has coordinates $(0, 0)$, and $C_1$, $C_2$ and $P$ are horizontally aligned, in that order from left to right, as shown in the following picture.

As $C_2$ rotates clockwise and moves anticlockwise around the inside of $C_1$, when $C_1C_2$ has gone from an angle of 0 to a positive angle of $\theta$, and $C_2P$ from an angle of 0 to a negative angle of $\psi$, the point of contact $T$ between both circles has travelled the same distance along both circles—represented in red in the picture below—, namely, $\theta R$ on $C_1$, and $(\theta - \psi)r$ on $C_2$. Hence:

$$\psi = -\frac{R - r}{r} \theta$$

At this stage, since $\overrightarrow{C_1P} = \overrightarrow{C_1C_2} + \overrightarrow{C_2P}$, the point $P$ has coordinates:

$$x = (R - r) \cos(\theta) + d \cos(-\psi)$$
$$y = (R - r) \sin(\theta) + d \sin(-\psi)$$

that is:

$$x = (R - r) \cos(\theta) + d \cos\left(\frac{R - r}{r} \theta\right)$$
$$y = (R - r) \sin(\theta) - d \sin\left(\frac{R - r}{r} \theta\right)$$

*Date: Session 2, 2016.*
Note that $P$ can “stick out” of $C_2$, that is, $d$ can be larger than $r$, as shown in the following picture, which also illustrates that $C_2$ can be larger than $C_1$, that is, $r$ can be greater than $R$; that does not change the above reasoning and the equations still hold.

The period of a hypotrochoid is the number of rolls of $C_2$ needed for $P$ to get back to its original position. It is equal to the least strictly positive integer $\rho$ such that $\rho \times 2\pi r$ is a multiple of $2\pi R$; hence it is equal to $\frac{r}{\gcd(r,R)}$.

2. Epitrochoids

If we let $C_2$ roll around the outside rather than the inside of $C_1$, then the curve obtained by tracing the positions taken by $P$ is called an *epitrochoid*. To compute the equation of the curve, one assumes that $C_1$, $C_2$ and $P$ are horizontally aligned, with $C_2$ to the right of $C_1$ and with $P$ to the left of $C_2$, and also to the left of $C_1$ in case $d$ is greater than $R + r$; the following picture illustrates the case where $r < R$ and $d < r$.

The reasoning that yields the equations for hypotrochoids can be immediately adapted to epitrochoids and result in the following equations:

\[
\begin{align*}
x &= (R + r) \cos(\theta) - d \cos\left(\frac{R + r}{r} \theta\right) \\
y &= (R + r) \sin(\theta) - d \sin\left(\frac{R + r}{r} \theta\right)
\end{align*}
\]

The period of an epitrochoid is also equal to $\frac{r}{\gcd(r,R)}$.

3. Particular cases

*Ellipse, deltoid, astroid, nephroid, cardioid* and *roses* are amongst the following pictures of epitrochoids (with a green filling) and hypotrochoids (with a yellow filling).
The following table shows how ellipse, deltoid, astroid, nephroid and a few other particular cases are obtained. When $d$ is equal to $r$, hypotrochoids are also called hypocycloids, and epitrochoids are also called epicycloids.

<table>
<thead>
<tr>
<th>Hypotrochoids</th>
<th>Epitrochoids</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r = \frac{R}{2}$</td>
<td>$r = \frac{R}{2}$, $\frac{2R}{3}$, $\frac{3R}{4}$</td>
</tr>
<tr>
<td>$d = r$ ellipse</td>
<td>deltoid astroid nephroid cardioid nephroid cardioid</td>
</tr>
<tr>
<td>$d = 0$ segment</td>
<td></td>
</tr>
<tr>
<td>Any $d$</td>
<td></td>
</tr>
</tbody>
</table>

To be complete, one should let $R$ be $\infty$; then $C_1$ is a line and the associated curves are called trochoids, with cycloids as a particular case when $d = r$...

COMP9021 Principles of Programming