# GSOE9210 Engineering Decisions

## Problem Set 04

1. At a school fête, a suspicious-looking man is offering bets on the toss of a coin of questionable fairness. The man is offering \$2 for each dollar bet (plus your original dollar) if the contestant chooses the face on the coin correctly.

   (a) Suppose Alice has a \$10 note in her pocket; represent the gamble as a decision table. Include the option of leaving (L) (*i.e.*, refusing to gamble).

   (b) If Alice were pessimistic (*i.e.*, she used *Maximin* as her decision rule), would Alice bet on heads or tails, or not bet?

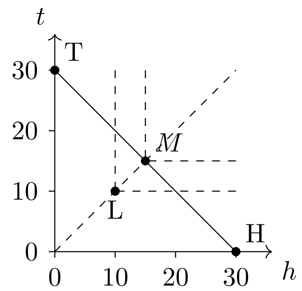
   (c) Suppose Alice has a similarly pessimistic friend Bob, and both could bet together on the same toss, would that affect Alice's decision?

   (d) Suppose Alice was friendless but had—instead of one \$10 note—10 \$1 coins in her pocket. How might this affect her decision?

   *Solution*

   (a) Assuming that one always has the option to not gamble, this situation is represented below, showing Alice's cash balance.

| | $h$ | $t$ |
|---|---|---|
| H | 30 | 0 |
| T | 0 | 30 |
| L | 10 | 10 |
| $\frac{1}{2}$H$\frac{1}{2}$T | $\frac{30}{2}$ | $\frac{30}{2}$ |




   (b) Alice would prefer to leave (L), guaranteeing that she keeps her \$10, otherwise she risks losing it, despite the prospect of winning \$20 (*i.e.*, a total balance of \$30)—Alice is *risk averse.*

   (c) If they each bet on opposite outcomes one of the two will win, so they are *guaranteed* \$30 between them. If they divide the balance, they would both finish with \$15, each doing better than not betting. Mixed actions can be thought of as assuming the decision-maker(s) are making multiple independent decisions on a single trial.

   (d) If Alice put puts \$5 ($5 \times \$1$) on H and \$5 ($5 \times \$1$) on T at those odds she would similarly guarantee herself a return of \$15 in total. An average return of \$1.5 per dollar bet.

2. (a) Find the value $\frac{3}{4}$ of the way from 4 to 2 on the number line.
   (b) Find a general expression for the value $\mu$ of the way from $a$ to $b$ on the number line.

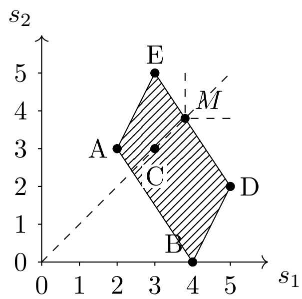
   Consider the decision problem below:

| | $s_1$ | $s_2$ |
|---|---|---|
| A | 2 | 3 |
| B | 4 | 0 |
| C | 3 | 3 |
| D | 5 | 2 |
| E | 3 | 5 |

   (c) Plot the actions as points on the Cartesian plane. Find the coordinates of the point $\mu$ of the way from A to B.
   (d) Find the point, $P(x, y)$, on the segment AB that intersects with the diagonal $y = x$.
   (e) Find the *Maximin* mixed action for the decision problem above.

*Solution*

| | $s_1$ | $s_2$ |
|---|---|---|
| A | 2 | 3 |
| B | 4 | 0 |
| C | 3 | 3 |
| D | 5 | 2 |
| E | 3 | 5 |




(a) The value is expressed as a convex combination with parameter $\mu$: $x = 2\mu + 4(1 - \mu) = 4 + (2 - 4)\mu = 4 - 2\mu$

(b) In general: $x = b\mu + a(1 - \mu) = a + (b - a)\mu$

(c) The point $P_\mu$, which is $\mu$ of the way from B$(4, 0)$ to A$(2, 3)$ is given by $P_\mu = (x, y)$, where $x$ is $\mu$ of the way from 4 to 2 and $y$ is $\mu$ of the way from 0 to 3 (by similar triangles). Hence:

$$x = 2\mu + 4(1 - \mu)$$
$$y = 3\mu + 0(1 - \mu)$$

Alternatively, in vector form:

$$(x, y) = \mu(2, 3) + (1 - \mu)(4, 0)$$

(d) Setting $x = y$ gives:

$$\begin{aligned} 2\mu + 4(1 - \mu) &= 3\mu + 0(1 - \mu) \\ 4 - 2\mu &= 3\mu \\ 5\mu &= 4 \\ \therefore\ \mu &= \tfrac{4}{5} \end{aligned}$$

(e) From the diagram, by inspection, the *Maximin* mixed action must correspond to a point lying on the segment DE. The point which is $\mu$ of the way from E$(3,5)$ to D$(5,2)$ is given by $P_\mu = (x,y)$, where

$$x = 5\mu + 3(1-\mu)$$
$$y = 2\mu + 5(1-\mu)$$

Setting $x = y$ gives:

$$\begin{aligned} 5\mu + 3(1-\mu) &= 2\mu + 5(1-\mu) \\ 2\mu + 3 &= -3\mu + 5 \\ 5\mu &= 2 \\ \therefore\ \mu &= \tfrac{2}{5} \end{aligned}$$

Here $\mu_{\mathrm{D}} = \mu = \frac{2}{5}$, and, hence, $\mu_{\mathrm{E}} = (1-\mu) = \frac{3}{5}$. So the *Maximin* mixed action is $\frac{2}{5}\mathrm{D}\frac{3}{5}\mathrm{E}$. Setting $\mu = \frac{2}{5}$ in the expressions for $x, y$ above, yields: $x = 5(\frac{2}{5}) + 3(\frac{3}{5}) = 2 + \frac{9}{5} = 3\frac{4}{5} = y$. Hence $P_\mu = (3\frac{4}{5}, 3\frac{4}{5}) = (3.8, 3.8)$. This is corroborated by the diagram.

3. Consider a scenario with four states, two pure actions, and their mixtures $(M)$, where $\mu_A = \mu$:

| | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|
| A | 4 | 2 | 1 | 3 |
| B | 0 | 1 | 5 | 2 |

(a) Draw the mixture plot for this problem.

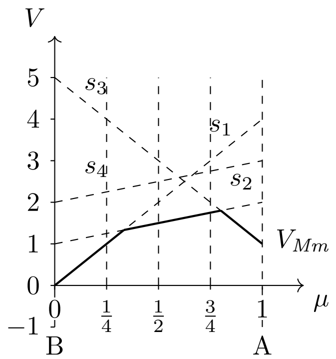
(b) Do all states need to be considered when determining the *Maximin* mixed action?

(c) Find the *Maximin* mixed action, and the *Maximin* value—*i.e.*, the value of the *Maximin* mixed action—for this problem.

*Solution*
The decision table, including a generic mixture, is shown below:

| | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|
| A | 4 | 2 | 1 | 3 |
| B | 0 | 1 | 5 | 2 |
| $M(\mu)$ | $4\mu$ | $\mu + 1$ | $5 - 4\mu$ | $\mu + 2$ |

(a) The mixture plot is shown below.




(b) Consider an arbitrary mixed action $M(\mu)$. From the plot above, the value in $s_4$ is always going to be greater than that in $s_2$, so the former will never feature in *Maximin* considerations. It follows that $s_4$ can be ignored with respect to *Maximin*.

(c) The *Maximin* mixed strategy is obtained by finding the value of $\mu$ for which the values in $s_2$ and $s_3$ match—noting, as we did above, that $s_2$ and $s_3$ are the only two relevant states, as they are responsible for the minimum values of each mixed action.
Setting the values in $s_2$ and $s_3$ to be equal gives:

$$\begin{aligned} \mu + 1 &= 5 - 4\mu \\ 5\mu &= 4 \\ \therefore \quad \mu &= \tfrac{4}{5} \end{aligned}$$

The *Maximin* mixed action is $M(\frac{4}{5})$.
The *Maximin* value is the value of the *Maximin* mixed action:

$$\begin{aligned} V_{Mm}(M(\tfrac{4}{5})) &= v(M(\tfrac{4}{5}), s_2) \\ &= \tfrac{4}{5} + 1 = \tfrac{9}{5} \end{aligned}$$

4. A drinks seller can purchase stock of several types of drink: a) hot chocolate; b) iced tea; c) lemonade; d) orange juice.

   She knows, from past experience, that on warm days she'll make sales totalling \$10 on hot chocolate, \$40 on iced tea, \$30 on lemonade, and \$40 on orange juice. On cool days, however, her sales total \$30 on hot chocolate, \$0 on iced tea, \$20 on lemonade, and \$10 on orange juice.

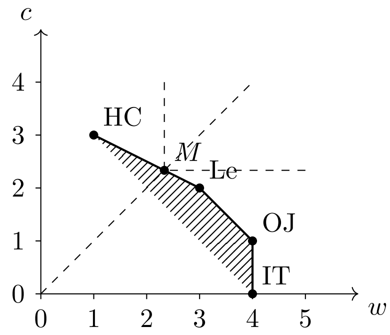
   (a) Produce a decision table for this problem.
   (b) What proportion of drinks should she stock to maximise her guaranteed sales total regardless of the temperature?
   (c) Draw the admissibility frontier for this problem. Are any actions inadmissible?

*Solution*

(a) Consider the decision table below. Values are expressed in tens of dollars. The associated graph is also shown.

| | $w$ | $c$ |
|---|---|---|
| HC | 1 | 3 |
| IT | 4 | 0 |
| Le | 3 | 2 |
| OJ | 4 | 1 |

where: $w$ warm day
$c$ cold day




(b) She would maximise her guaranteed sales by having the mixture of stock which maximises the minimum sales irrespective of whether the day is warm or cold.
It is clear from the graph that the optimal mixture should comprise hot chocolate and lemonade only.
Let $m_w$ be the average sales of the relevant mixture of drinks on a warm day and $m_c$ the mixture's average sales on a cool day.
If $\mu$ is the desired proportion of hot chocolate in the mixture, then $M = (m_w, m_c) = (3, 2) + \mu[(1, 3) - (3, 2)]$; *i.e.*,

$$\begin{aligned} m_w &= 3 + (1-3)\mu = 3 - 2\mu \\ m_c &= 2 + (3-2)\mu = 2 + \mu \end{aligned}$$

Setting $m_w = m_c$ to find the *Maximin* mixed strategy:

$$\begin{aligned} 3 - 2\mu &= 2 + \mu \\ 1 &= 3\mu \\ \therefore \quad \mu &= \tfrac{1}{3} \end{aligned}$$

That is, she should have a mixture consisting of one third of the units on sale being hot chocolate and the other two thirds lemonade.
Note that this is a ratio of two units of lemonade per unit of hot chocolate.

(c) The admissibility frontier is marked on the graph as the solid line connecting HC, Le, OJ, and IT. All pure strategies are admissible, provided we consider strict dominance. Under weak dominance IT would be regarded inadmissible.

5. Alice is considering whether to invest \$1000 over an investment period, and if so on which option to invest. She is looking at an investment market which will either rise ($r$) in value by 6% or flatten ($f$) to 0% over the investment period. She will learn the market movement mid-period, at which point she can choose to invest in a fixed rate option (F), which gives a constant return of 2%, or take a risky option (R) which follows the market's movement.

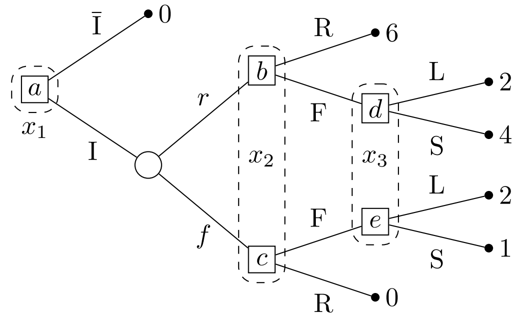
For the fixed option, Alice has the added option to invest initially for a long term (L) (*i.e.*, for the full period), or a short term (S) (half period). If she invests short term she will respond to the market in either of two ways: (a) if the market has risen she'll change to the risky option, earning an average profit of 4%; (b) if the market has fallen she'll reinvest in F but will receive a lower rate, which would reduce her gains to 1% per annum.

She must submit her investment instructions ($\bar{\text{I}}$, R, or F, and for the latter, L or S) to her stock broker before the market's movement is known.

(a) Represent this situation as a decision tree (*i.e.*, in extensive form) and as a decision table.
(b) How many information sets are there in this problem?
(c) Assuming diversified stock portfolios (*i.e.*, mixtures of investments) are allowed, which mixed strategies are admissible?
(d) Which is the *Maximin* mixed strategy?
(e) Which is the *miniMax Regret* mixed strategy?

*Solution*

(a) One possible extensive form is shown below. The additional symbols are: I: invest; $\bar{\text{I}}$: don't invest.

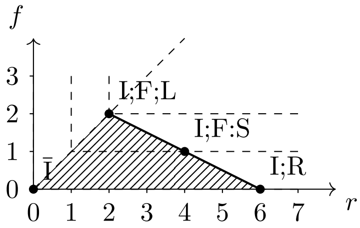



The decision table is shown below. Inadmissible (dominated) pure strategies have been struck out.

| | $r$ | $f$ |
|---|---|---|
| ~~$\bar{\text{I}}$~~ | ~~0~~ | ~~0~~ |
| I;F;L | 2 | 2 |
| I;F:S | 4 | 1 |
| I;R | 6 | 0 |

(b) There are three information sets in total: $x_1 = \{a\}$, $x_2 = \{b, c\}$, and $x_3 = \{d, e\}$. Notice also that, because they are indistinguishable to the agent, the nodes in each information have the same number of identically labelled branches.

(c) Graphically:




Admissible mixed strategies are represented by points on the solid line. None of these strategies are better than any other in all (both) cases/states. All other mixed strategies are inadmissible.

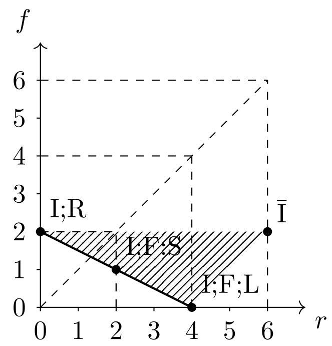
(d) It is clear from the graph than I;F;L is the *Maximin* mixed strategy. In this case, it happens to be a pure strategy.

(e) In terms of regret:

| | $r$ | $f$ |
|---|---|---|
| ~~Ī~~ | ~~6~~ | ~~2~~ |
| I;F;L | 4 | 0 |
| I;F:S | 2 | 1 |
| I;R | 0 | 2 |

Note that for regret we prefer smaller values, so a strategy is dominated if its values are greater than those of another action, as indicated above.

The regret graph is shown below:




Notice that the *miniMax Regret* pure strategies are I;F:S and I;R, whereas the *Maximin* pure strategy is I;F;L.

Would a person who wishes to guarantee the best possible return on her investment think in terms of value or regret?

Moreover, if the market was unpredictable, and the \$1000 can be divided among multiple investments (*e.g.*, as in an investment portfolio), then 'hedging', by investing in a mixture of I;R and I;F:S, would be best to minimise the maximum regret.

6. Consider the following decision table, in which mixed strategies are allowed.

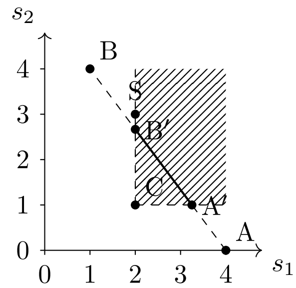
| | $s_1$ | $s_2$ |
|---|---|---|
| A | 4 | 0 |
| B | 1 | 4 |
| C | 2 | 1 |

(a) Which, if any, strategies are dominated?

(b) Prove that a possible strategy S, with payoffs 2 and 3 in states $s_1$ and $s_1$ respectively, would not be dominated.

*Solution*

(a) Strategy C is dominated by a mixture of A and B.




| | $s_1$ | $s_2$ |
|---|---|---|
| A | 4 | 0 |
| B | 1 | 4 |
| ~~C~~ | ~~2~~ | ~~1~~ |

The reduced matrix is shown above (right).

(b) Consider mixtures of A and B with parameter $\mu$ representing the proportion of A in the mixture; *i.e.*, $M(\mu) = \mu\text{B} + (1-\mu)\text{B}$.
It follows that $M(\mu) = (3\mu + 1, 4 - 4\mu)$. For any mixture $M(\mu)$ to strictly dominate S we require that $3\mu + 1 > 2$ and $4 - 4\mu > 3$; *i.e.*, $\mu > \frac{1}{3}$ and $\mu < \frac{1}{4}$. Since there is no $\mu$ that can simultaneously satisfy both conditions, it follows that there can be no mixture of A and B that dominates S.