GSOE9210 Engineering Decisions

Problem Set 06

1. Alice has \$1000 and has been offered a gamble represented by the lottery:

$$\ell = [\tfrac{1}{3} : \$4000 | \tfrac{2}{3} : \$0]$$

   (a) What is the expected monetary value (EMV) of the gamble?
   (b) If it costs \$1000 to gamble, would it be a fair bet?
   (c) If Alice prefers to not bet, for this bet she is: a) risk averse; b) risk seeking; or c) risk neutral?
   (d) Describe the shape of Alice's utility curve.

   Solution

   (a) The EMV is the *Bayes* value assuming $v = v_\$$; *i.e.*, using the dollar amount as the payoff.

$$\begin{aligned} V_\$(\ell) = E(v_\$) &= \tfrac{1}{3} v_\$(\$4000) + \tfrac{2}{3} v_\$(\$0) \\ &= \tfrac{1}{3}(4000) + \tfrac{2}{3}(0) \qquad \text{because } v_\$(\$x) = x \\ &= \tfrac{1}{3}(4000) = 1333 \end{aligned}$$

   (b) A bet is fair for Alice if she would expect to be no worse off by gambling; *i.e.*, the gamble's EMV is greater than her initial amount. Because the gamble's EMV is \$1333, which is greater than her initial amount (\$1000), the bet is fair for Alice.
   (c) Alice is risk averse because she would not choose to gamble on a fair bet; *i.e.*, a bet whose expected monetary value is greater than her initial amount.
   (d) Her utility curve is (likely to be) concave down over the region \$0 to \$4000.

2. Consider the example in lectures in which Alice was indifferent between a certain \$10 and the lottery $[\frac{1}{2} : \$50 | \frac{1}{2} : \$0]$. Based on Alice's utility function, as given in lectures, estimate Alice's certainty equivalent for the lottery in which she shares her profits with Bob. Use this to determine the risk premium for the combined gamble.

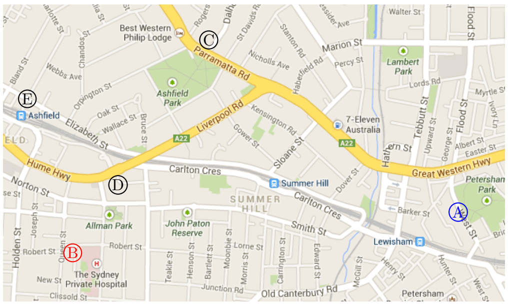
   Solution
   By inspection, the certainty equivalent is about \$13. Since the EMV of the lottery is about \$20, the risk premium is about $\$7 = \$20 - \$13$.

3. An agent with a non-decreasing utility function who is: a) risk averse; b) risk seeking; or c) risk neutral, has risk premium in what range?

Solution

An agent with a non-decreasing utility function has: a) a positive risk premium if she's risk averse; b) a negative risk premium if she's risk seeking; or c) a zero risk premium if she's risk neutral.

4. Consider the travelling problem from lectures:



	b_L	b_P
Tr	20	20
Bu	10	40

where payoffs represent travel times in minutes.

A surgeon needs to travel from A to B to treat a patient suffering from an aggressive bacterial infection. If the infection isn't treated it will start to cause damage after 15 minutes. Death, due to organ failure, will result after 40 minutes.

(a) Suppose the surgeon would consider taking the train (travel time: 20 minutes) over the bus if there is only a 10% chance of the bus going down Liverpool Road (*i.e.*, travel time: 10 minutes). Draw the decision table in terms of utility values.

(b) What might the surgeon's utility for (*i.e.*, as a function of) time look like?

(c) For what range of probabilities of the bus going down Liverpool Road would the surgeon prefer to take the bus?

(d) Suppose the surgeon has the option of driving. Driving is subject to traffic and would take 15 minutes in light traffic and twice as long in heavy traffic. Suppose that the surgeon believes there's a 40% chance of heavy traffic. Using the utility graph of part b), or otherwise, estimate the surgeon's risk premium (in terms of travel time) for driving.

Solution

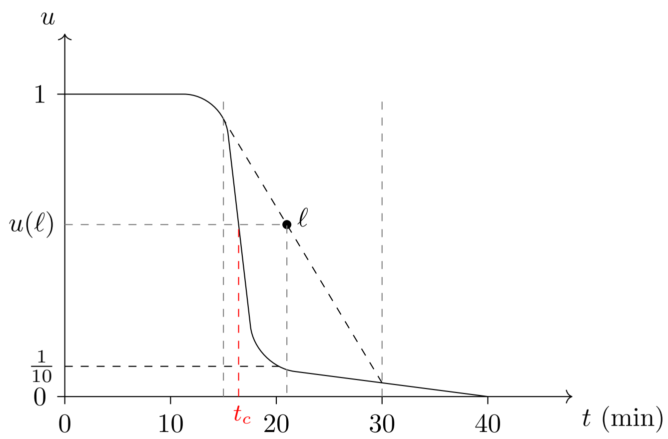
(a) Setting $u(10\text{min}) = 1$ and $u(40\text{min}) = 0$:

$$u(20\text{min}) = \tfrac{1}{10}u(10\text{min}) + \tfrac{9}{10}u(40\text{min}) = \tfrac{1}{10}$$

So the decision table in terms of utility is:

	b_L	b_P
Tr	0.1	0.1
Bu	1	0

(b) You would expect the utility would be close to 1 (the patient is treated with no major damage) for times less than the 15 minute mark, and drop off steeply to a utility close to 0 by the 20 minute mark, based on the surgeon's preferences (*i.e.*, indifferent between 20 minutes and a 10% change of 10 minutes). A possible utility function for the surgeon might be as follows:



(c) The *Bayes* utilities are:

$$\begin{aligned} U(\text{Tr}) &= u(20\text{min}) = 0.1 \\ U(\text{Bu}) &= pu(10\text{min}) + (1-p)u(40\text{min}) \\ &= p \end{aligned}$$

The surgeon would prefer to take the bus if it has a higher utility than taking the train; *i.e.*, $U(\text{Bu}) > U(\text{Tr})$.
This is the case if $p > \frac{1}{10}$; *i.e.*, if the probability of the bus going down Liverpool Rd is greater than $\frac{1}{10}$.

(d) Driving corresponds to the lottery $\ell = [\frac{6}{10} : 15 \text{ min}|\frac{4}{10} : 30 \text{ min}]$. This lottery is plotted in the graph above. From the graph, the certainty equivalent time t_c of driving is approximately 17 minutes.
The *Bayes* value using time is $V_B(\ell) = \frac{6}{10}(15) + \frac{4}{10}(30) = 9 + 12 = 21$ minutes. So the risk premium, in terms of time, is $21 - 17 = 4$ minutes. In this case, because lesser times are more preferred (*i.e.*, the utility function is a decreasing function of time), this would mean that the surgeon 'over-values' driving (matching it, in terms of her preferences, with 17 minutes, compared to the expected travel time of 21 minutes); *i.e.*, in this case, a positive risk premium means the surgeon is risk-seeking (she prefers to take risks to prevent any damage).

5. Consider a decision problem with the following decision table for options A, B, and C, where the outcomes are in dollar amounts and the states' probabilities are shown above the respective states:

	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
	s_1	s_2	s_3
A	20	30	40
B	10	20	40
C	20	30	20

(a) If the agent were risk neutral, which option should she choose?

(b) Suppose now that agent's preferences were such that she would be indifferent between \$20 and the lottery $[\frac{1}{2} : \$10 | \frac{1}{2} : \$40]$, and between \$30 and the lottery $[\frac{1}{5} : \$10 | \frac{4}{5} : \$40]$.
Which of A, B, or C would the agent choose?

(c) For part b), is the agent risk neutral, averse, or prone/seeking?

(d) Estimate the agent's certainty equivalent and risk premium for A.

Solution

(a) For a risk neutral agent we can use monetary values:

$$\begin{aligned}
V_{\$}(\text{A}) &= \tfrac{1}{4}(\$20) + \tfrac{1}{2}(\$30) + \tfrac{1}{4}(\$40) \\
&= \$5 + \$15 + \$10 \\
&= \$30 \\
V_{\$}(\text{B}) &= \tfrac{1}{4}(\$10) + \tfrac{1}{2}(\$20) + \tfrac{1}{4}(\$40) \\
&= \$2.5 + \$10 + \$10 \\
&= \$22.5 \\
V_{\$}(\text{C}) &= \tfrac{1}{4}(\$20) + \tfrac{1}{2}(\$30) + \tfrac{1}{4}(\$20) \\
&= \$5 + \$15 + \$5 \\
&= \$25
\end{aligned}$$

Therefore the agent should choose A.
Alternatively, by dominance considerations, A weakly dominates the other choices. Note that provided all probabilities involved are non-zero, weak dominance is enough to guarantee that an action will be strictly preferred under the *Bayes* decision rule (see Exercise Set 05).

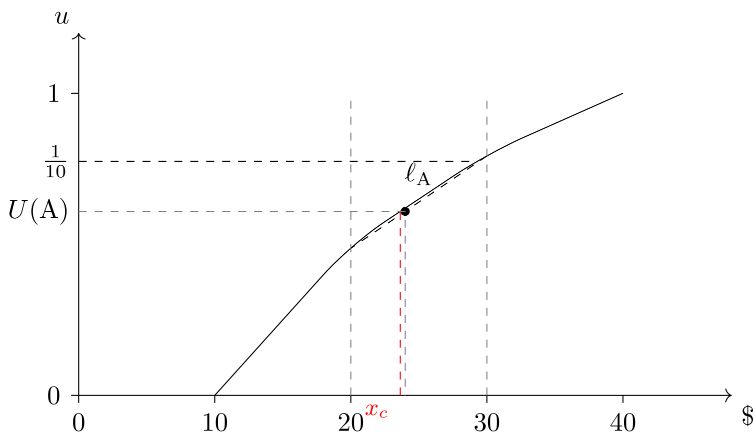
(b) Note that the lotteries given are standard (reference) lotteries. Setting $u(\$10) = 0$ and $u(\$40) = 1$. We get $u(\$20) = 0.5$ and $u(\$30) = 0.8$.
Based on these utilities we obtain the table:

	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
	s_1	s_2	s_3
A	0.5	0.8	1
B	0	0.5	1
C	0.5	0.8	0.5

For money, the utility function must be non-decreasing, and hence dominance is preserved when moving to utilities.

$$\begin{aligned}
U(\mathrm{A}) &= \tfrac{1}{4}u(\$20) + \tfrac{1}{2}u(\$30) + \tfrac{1}{4}u(\$40) \\
&= \tfrac{1}{4}(0.5) + \tfrac{1}{2}(0.8) + \tfrac{1}{4}(1) \\
&= 0.775 \\
U(\mathrm{B}) &= \tfrac{1}{4}u(\$10) + \tfrac{1}{2}u(\$20) + \tfrac{1}{4}u(\$40) \\
&= \tfrac{1}{4}(0) + \tfrac{1}{2}(0.5) + \tfrac{1}{4}(1) \\
&= 0.5 \\
U(\mathrm{C}) &= \tfrac{1}{4}u(\$20) + \tfrac{1}{2}u(\$30) + \tfrac{1}{4}u(\$20) \\
&= \tfrac{1}{4}(0.5) + \tfrac{1}{2}(0.8) + \tfrac{1}{4}(0.5) \\
&= 0.65
\end{aligned}$$

(c) Consider the graph below:



Notice that $V_\$(\mathrm{A}) = \$30$, whereas $U(\mathrm{A}) = 0.775$. But $u(\$30) = 0.8$. Therefore $u(V_\$(\mathrm{A})) = u(\$30) = 0.8 \geqslant 0.775 = U(\mathrm{A})$; *i.e.*, the agent is slightly risk averse.

As an exercise, verify the same for B and C.

(d) Since $0.5 < U(\mathrm{A}) < 0.8$, linearly interpolating to estimate u over this interval, set:

$$\begin{aligned}
\frac{x-20}{30-20} &\approx \frac{u-0.5}{0.8-0.5} \\
\frac{x-20}{10} &= \frac{u-0.5}{0.3} \\
3(x-20) &= 100(u-0.5) \\
\therefore \quad u(x) &\approx \tfrac{3}{100}(x-20) + 0.5 \\
&= \tfrac{3}{100}x - 0.1
\end{aligned}$$

The certainty equivalent x_c is determined by: $u(x_c) = U(\mathrm{A}) = 0.775$.

This gives:

$$\begin{aligned}
\tfrac{3}{100}x_c - 0.1 &= 0.775 \\
\tfrac{3}{100}x_c &= 0.875 \\
x_c &= \tfrac{100}{3}(0.875) = \tfrac{87.5}{3} \\
&\approx 29
\end{aligned}$$

Hence the certainty equivalent is approximately \$29. The risk premium is given by $V_{\$}(\text{A}) - x_c \approx \$30 - \$29 = \$1 \geqslant 0$.
If faced with such an option, the agent would be willing to forgo up to \$1 of the expected gain to avoid the risk. The agent is only mildly risk averse.

6. A *two-way winner-takes-all* gamble between two participants A and B is a gamble in which, if the two participants bet amounts $\$a$ and $\$b$ respectively, the winner takes everything ($\$(a+b)$). Let p be the probability that A wins. Prove that a two-way winner-takes-all bet is fair to both participants iff:
$$\frac{a}{b} = \frac{p}{1-p}$$

   Solution
   For a fair bet $p(a+b) + (1-p)0 = a$. Solving for p: $p = \frac{a}{a+b}$. But then $1 - p = \frac{b}{a+b}$. Therefore:
$$\frac{a}{b} = \frac{p}{1-p}$$

7. Which of the following real-valued functions are: a) non-decreasing; b) strictly increasing; c) one-to-one; d) onto.
$$f(x) = x, f(x) = x + 1, f(x) = x^2, f(x) = |x|$$

   Solution

   - $f(x) = x$ is strictly increasing (hence monotonic increasing/non-decreasing), one-to-one, and onto
   - $f(x) = x + 1$ is all of the above
   - $f(x) = x^2$ fails all of the above over all of $\mathbb{R}$ but is strictly increasing over $x \geqslant 0$; it is onto $\{y \in \mathbb{R} \mid y \geqslant 0\}$.
   - $f(x) = |x|$ has the same properties as $f(x) = x^2$.

8. Show that for a non-decreasing function it is not true in general that if $f(x) \geqslant f(y)$, then $x \geqslant y$.

   Solution
   Consider the constant function $f(x) = c$, for fixed $c \in \mathbb{R}$. It is non-decreasing as for any x, y, $f(x) = f(y) = c$, hence $f(x) \geqslant f(y)$. However, consider any $x, y \in \mathbb{R}$ such that $y > x$. Then $f(x) \geqslant f(y)$, but $x \not\geqslant y$.

9. Show that for any real-valued function $f : \mathbb{R} \to \mathbb{R}$, the following are equivalent:

(a) f is strictly increasing
(b) for any $x, y \in \mathbb{R}$, $x > y$ iff $f(x) > f(y)$
(c) for any $x, y \in \mathbb{R}$, $x \geqslant y$ iff $f(x) \geqslant f(y)$

Solution
Prove that (a) implies (b) which implies (c) which implies (a) again, this cyclical sequence of implications entails that all three are logically equivalent.