GSOE9210 Engineering Decisions

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Utility theory

Outline

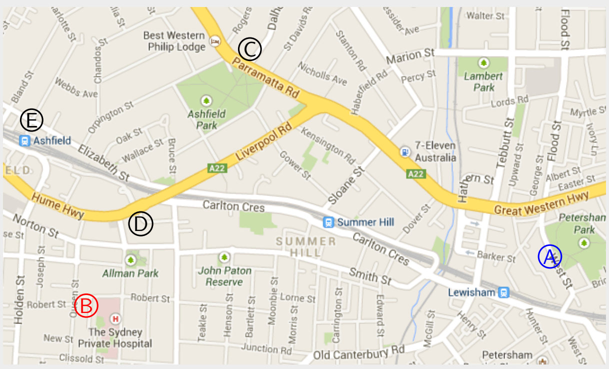
1. Utility theory
   - Preference relations
   - Ordering prizes/outcomes
   - Evaluating prizes
   - Evaluating lotteries

Evaluating outcomes and actions

Example (Bus or train?)

Would Alice prefer to catch the bus or the train if:

1. she's a doctor on an emergency call
2. has an injured foot
3. is a tourist.



- How to compare outcomes: travel time, walking distance, scenic appeal, comfort, *etc.*?
- How do we measure/quantify scenic appeal, comfort?

Preference and numbers

- So far preference based on numerical values assigned to outcomes and actions (*i.e.*, on v and V respectively); *i.e.*, an agent prefers:
  - outcome ω_1 to ω_2 if $v(\omega_1) > v(\omega_2)$
  - action A to B if $V(\mathrm{A}) > V(\mathrm{B})$
- Does value (which?) determine preference or preference determine value?
- Can meaningful numbers always be assigned? *e.g.*, Alice is a tourist who values comfort and good scenery
- Can rational decisions be made when numerical values aren't given/available?

Preference and numbers

- Numbers aren't always required; consider the *Maximin* rule:

	s_1	s_2
A	v_{11}	v_{12}
B	v_{21}	v_{22}

	s_1	s_2
A	20	0
B	16	8

	s_1	s_2
A	9	2
B	8	3

- *Maximin* is independent of specific values assigned to outcomes, provided *preference order* is preserved: *i.e.*, $v_{11} > v_{21} > v_{22} > v_{12}$

Exercise

Will this be still be the case for *Hurwicz*'s rule ($\alpha = \frac{1}{4}$)? *miniMax Regret*? Laplace's rule?

Qualitative preference: preference without numbers

- *Maximin* can be reformulated in terms of *qualitative preferences* only

	s_1	s_2
A	ω_{11}	ω_{12}
B	ω_{21}	ω_{22}

Preferences
ω_{11} preferred to ω_{21}
ω_{21} preferred to ω_{22}
ω_{22} preferred to ω_{12}

Definition (Qualitative *Maximin*)

Associate an action with its/a least preferred outcome. Choose action whose associated outcome is most preferred.

- Which is least preferred outcome of A? *i.e.*, ω_{11} preferred to ω_{12}?

Preference and value

Consequences of assigning numerical quantities (*i.e.*, via some value function $v : \Omega \to \mathbb{R}$) to encode preference:

- agent either prefers a to b, or b to a, or agent prefers them equally—agent *indifferent* between a and b
- if agent prefers a to b, and b to c, then agent prefers a to c; *i.e.*, preferences *transitive*

Questions

- Are these conditions justified in practice?
- Do actual (human) agents always behave in this way?
- Can you find counter-examples?

Consistent preferences

- Rational decisions can be made without numerical values so long as an agent's *preferences* are 'consistent'
- What does 'preference consistency' mean?

	s_1	s_2
A	ω_{11}	ω_{12}
B	ω_{21}	ω_{22}

Preferences
ω_{11} preferred to ω_{21}
ω_{21} preferred to ω_{22}
ω_{22} preferred to ω_{12}

- Then, for example:
  - ω_{11} preferred to ω_{12}
  - ω_{21} not preferred to ω_{11}

Preference consistency

- A rational agent's (strict) preferences should be consistent in the sense that, *e.g.*, an agent that:
  - prefers apples (A) to bananas (B) shouldn't prefer bananas to apples
  - prefers apples (A) to bananas (B) and bananas (B) to carrots (C) shouldn't prefer carrots (C) to apples (A)

Exercises

- What would be consequences of the failure of the first property above?
- In the second property above, should the agent then necessarily prefer apples to carrots?

- Preferences is a *binary relations*

Binary relations: overview

Modelling binary relations:

- If A and B are sets, define the *Cartesian product* of A and B: $A \times B = \{(a, b) \mid a \in A \quad \& \quad b \in B\}$; *e.g.*, the set of all coordinate pairs on the Euclidean plane $\mathbb{R} \times \mathbb{R}$

Definition (Binary relation)

A *binary relation* R from A to B is a subset of $A \times B$; *i.e.*, $R \subseteq A \times B$. Each ordered pair $(x, y) \in R$ is called an *instance* of R.

- In *infix notation*: aRb iff $(a, b) \in R$; *e.g.*, $3 \leqslant 5$
- If aRb (*i.e.*, $(x, y) \in R$) then the relation R is said to *hold* for x with y; *e.g.*, because $3 \leqslant 5$, then $\leqslant$ holds for 3 with 5

Binary relations

Definition (Binary relation on a set A)

A binary relation, R, *on* a set A is a subset of $A \times A$; *i.e.*, $R \subseteq A \times A$.

- *e.g.*, the binary relation 'is greater than', written $> \subseteq \mathbb{R} \times \mathbb{R}$, is a binary relation on the set of real numbers $\mathbb{R}$ (and on $\mathbb{N}$, and on $\mathbb{Q}$)
- *e.g.*, the 'greater than' relation ($>$) holds between real numbers 3 and π (written $3 > \pi$); *i.e.*, $3 > \pi$ is an instance of $>$

Representing relations

- Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$, then a relation $R \subseteq A \times B$ can be represented by the matrix/table:

$$\begin{array}{cc|cccc} & & & B & & \\ & R & b_1 & b_2 & b_3 & b_4 \\ \hline & a_1 & & \times & \times & \\ A & a_2 & \times & \times & \times & \times \\ & a_3 & \times & & & \times \end{array}$$

- An $\times$ appears at entry in row x and column y iff xRy. More succinctly, there's a $\times$ at (x, y) iff xRy iff (x, y) is an instance of R. *e.g.*, above a_1Rb_2, a_2Rb_1, and a_3Rb_4, but $a_1 \not R b_1$.

Relational properties

Let R be a binary relation on some set A:

- R is *reflexive* iff for every $x \in A$, xRx; *e.g.*, for every $x \in \mathbb{R}$, $x = x$, $x \leqslant x$, $x \geqslant x$
- R is *irreflexive* iff for every $x \in A$, xRx does not hold; *e.g.*, for every $x \in \mathbb{R}$, $x \neq x$, $x < x$, $x > x$ do not hold
- R is *transitive* iff for any $x, y, z \in A$, when xRy and yRz, then xRz; *e.g.*, $=, <, \leqslant$ on $\mathbb{R}$
- R is *symmetric* iff for any $x, y \in A$, when xRy, then yRx; *e.g.*, $=$ on $\mathbb{R}$
- R is *total* iff xRy or yRx; *e.g.*, $=$, $\leqslant$ on $\mathbb{R}$
- R is *asymmetric* iff whenever xRy then yRx does not hold; *e.g.*, $<$ on $\mathbb{R}$
- R is *antisymmetric* iff whenever xRy and yRx, then $x = y$; *e.g.*, $\leqslant$ on $\mathbb{R}$

Preference relations

- A binary relation can be used to model *strict preference*:

Definition

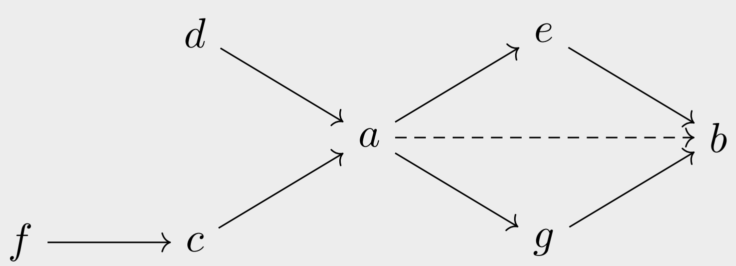
An agent *strictly prefers* element a to b, written $a \succ b$, iff it prefers a more than b; *i.e.*, it would eliminate b. The collection of all such instances comprises the agent's *strict preference relation*, $\succ$.

- What intuitive properties should strict preference relations satisfy?
  - if $x \succ y$, then it should not be the case that $y \succ x$
  - if $x \succ y$ and $y \succ z$, then it should not be the case that $z \succ x$

Representing preference: Hasse diagrams

- Suppose we're given the following strict preferences on a set $A = \{a, b, c, d, e, f, g\}$:

$$\begin{array}{llll} a \succ g & f \succ c & d \succ a & a \succ e \\ e \succ b & c \succ a & g \succ b & \end{array}$$



- Do we *know* that $a \succ b$? What about $c \succ d$? $f \succ d$? $e \succ g$?
- $x \succ y$ iff there's a path following arrows from x to y

Indifference: equal preference

Definition (Indifference)

If two elements a and b are *equally preferred* then the agent is said to be *indifferent* between them, written $a \sim b$. The set of all such instances constitutes an agent's binary relation of *indifference*. The *indifference class* of a is $[a] = \{b \mid a \sim b\}$.

Definition (Weak preference)

Element a is *weakly preferred* to b, written $a \succsim b$, iff a is strictly preferred to b or the two are equally preferred; *i.e.*, a is at least as preferred as b; *i.e.*, $a \succsim b$ iff $a \succ b$ or $a \sim b$.

Representing preference: Hasse diagrams

- Suppose we're given the following strict preferences on a set $A = \{a, b, c, d, e, f, g\}$:

$$\begin{array}{cccc} a \succ g & f \succ c & d \succ a & a \succ e \\ e \succ b & c \succ a & g \succ b & \end{array}$$

$$\begin{tikzcd}
& d \arrow[dr] & & e \arrow[dr] & \\
& & a \arrow[ur] \arrow[dr] \arrow[rr, dashed] & & b \\
f \arrow[r] & c \arrow[ur] & & g \arrow[ur] &
\end{tikzcd}$$

- Do we *know* that $a \succ b$? What about $c \succ d$? $f \succ d$? $e \succ g$?
- $x \succ y$ iff there's a path following arrows from x to y

Indifference properties

The following are intuitive properties of indifference:

- if $x \sim y$, then $y \sim x$
- if $x \sim y$ and $y \sim z$, then $x \sim z$
- $x \sim x$ holds for any $x \in A$

Combined properties:

- if $x \sim y$ and $z \succ x$, then $z \succ y$
- if $x \sim y$ and $x \succ z$, then $y \succ z$

Note that, in the previous problem, it would be *inconsistent* for $c \sim d$ and $f \sim d$, as $f \succ c$, which would imply $f \succ d$.

Axiomatisation of consistent weak preference

- What does it mean for an agent's preferences to be consistent/rational?
- Regard $\succsim$ as the fundamental/primitive notion, and interpret $x \succsim y$ as "x is at least as preferred as y"
- The following axioms characterise *consistent preference*

Axiom 1: Transitivity

The relation $\succsim$ is transitive; *i.e.*, preference accumulates.

Axiom 2: Comparability

The relation $\succsim$ is total; *i.e.*, every outcome is comparable.

Derived definitions

From the basic definition of $\succsim$ we can define indifference and strict preference as *derived notions*:

Definition (Indifference)

The relation of *indifference*, denoted $\sim$, is defined by:

$$x \sim y \text{ iff } x \succsim y \ \ \& \ \ y \succsim x.$$

Definition (Strict preference)

The relation of *strict preference*, denoted $\succ$, is defined by:

$$x \succ y \text{ iff } y \succsim x \text{ does not hold.}$$

Ordinal value functions

Definition (Ordinal value function)

An *ordinal value function* on a 'preference set' $(A, \succsim)$ is a function $v : A \to \mathbb{R}$ such that $v(x) \geqslant v(y)$ iff $x \succsim y$.

Exercise

Show that for any ordinal value function v:

- $v(x) > v(y)$ iff $x \succ y$
- $v(x) = v(y)$ iff $x \sim y$

Theorem (Consistency)

For any consistent preference relation there exists an ordinal value function.

Preference relations

Properties

The following properties follow from the earlier definitions:

- If an agent's preferences are consistent then $\sim$ is an equivalence relation
- The corresponding strict preference relation $\succ$ is a strict total order
- Strict preference satisfies an indifference version of the trichotomy law *i.e.*, exactly one of the following holds between any $x, y \in A$: $x \succ y$ or $x \sim y$ or $y \succ x$.

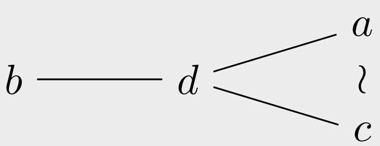
Weak preference

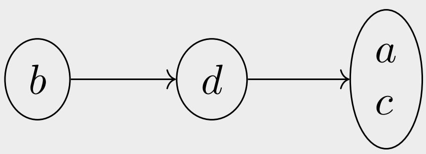
- Consider the following *complete* listing of weak preferences on a set $A = \{a, b, c, d\}$:

$$\begin{array}{lllll} a \succsim c & c \succsim a & b \succsim d & d \succsim a & d \succsim c \\ a \succsim a & c \succsim c & b \succsim b & d \succsim d & \\ b \succsim a & b \succsim c & & & \end{array}$$

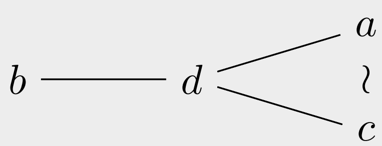
$$a \succsim c \quad c \succsim a \quad b \succsim d \quad d \succsim a \quad d \succsim c$$

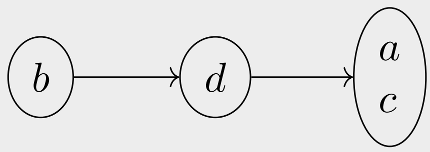
$$a \sim c \quad b \succ d \quad d \succ a \quad d \succ c$$





Generating rankings



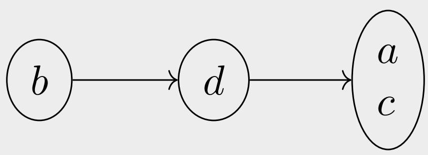


$\succ$	a	b	c	d
a				
b	$\times$		$\times$	$\times$
c				
d	$\times$		$\times$	

$\succ_I$	$\{b\}$	$\{d\}$	$\{a,c\}$
$\{b\}$		$\times$	$\times$
$\{d\}$			$\times$
$\{a,c\}$			

- The *rank* of x is $r(x) =$ number of $\times$in x's row; *e.g.*, $r(b) = 2, r(d) = 1$, and $r(a) = r(c) = 0$.
- This ranking is an ordinal value function

Ranking



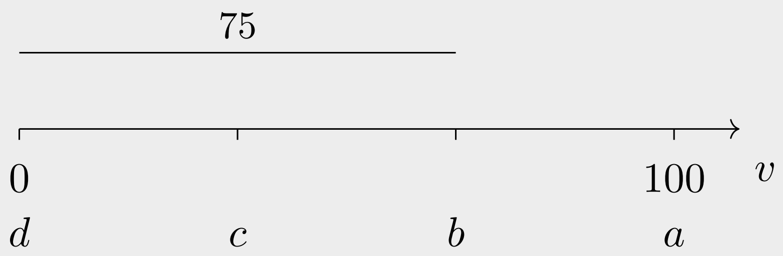
Definition (Rank)

The *rank* of an indifference is defined by the successive values assigned to indifference class when the lowest indifference class is assigned rank 0.

i.e., the ranks above are $0, 1, 2, \ldots$

Evaluating intermediate prizes

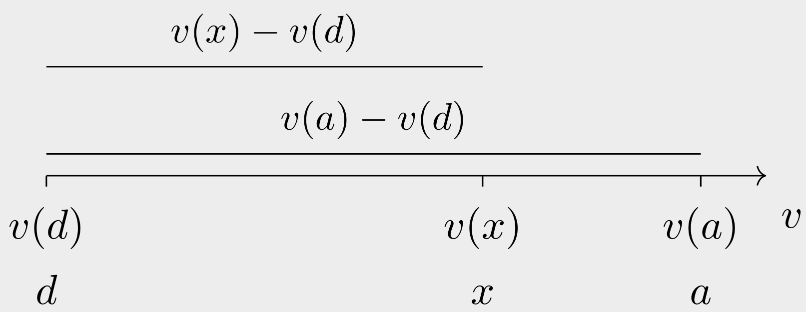
- Suppose the prizes in a lottery ℓ have been ordered by preference: $a \succ b \succ c \succ d$.
- Choose fixed reference values for the best and worst prizes, a and d: *e.g.*, $v(a) = 100$ and $v(d) = 0$



- Which values should be assigned to b? $100 \times rank(b)/rank(a)$?
- Agent's preferences: $b \sim [\frac{3}{4} : a|\frac{1}{4} : d]$
- Then $v(b)$ should be $V_B([\frac{3}{4} : a|\frac{1}{4} : d])$; *i.e.*, $v(b) = \frac{3}{4} \times 100 = 75$

Utility theory | Evaluating prizes

Evaluating intermediate prizes



In general, for prize x such that $x \sim [p_x : a|(1 - p_x) : d]$, for $0 \leqslant p_x \leqslant 1$, assign value $v(x)$, where:

$$\frac{v(x) - v(d)}{v(a) - v(d)} = p_x$$

i.e., $v(x) = \alpha p_x + \beta$, where $\alpha = v(a) - v(d)$ and $\beta = v(d)$

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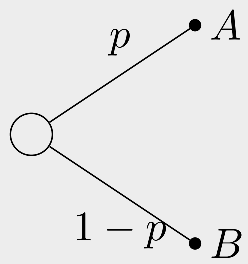
Binary lotteries

Definition (Binary lottery)

A *binary lottery* is a lottery in which at most two possible prizes have non-zero probability: *i.e.*, of the form $\ell = [p : A|(1-p) : B]$.



e.g., the lottery for tossing a fair coin: $\ell = [\frac{1}{2} : \mathsf{h}|\frac{1}{2} : \mathsf{t}]$.

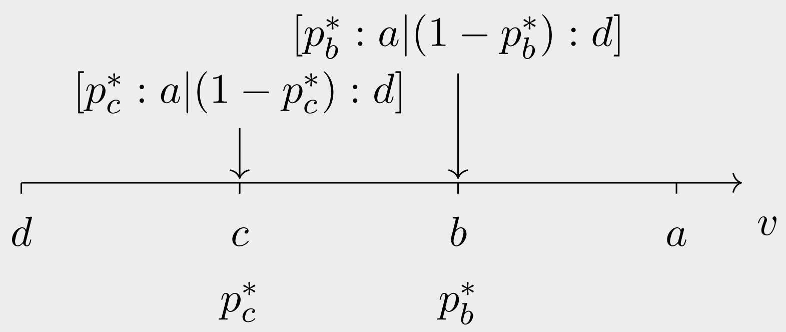
Reference lotteries

Definition (Reference lottery)

Let ω_M and ω_m be, respectively, the best and worst possible prizes ($\omega_M \succ \omega_m$). A *reference lottery*, ℓ^*, is a binary lottery:

$$\ell^* = [p : \omega_M|(1-p) : \omega_m]$$

If prize $x \sim \ell^*_x = [p^*_x : \omega_M|(1-p^*_x) : \omega_m]$, then ℓ^*_x is called the *reference lottery* for x, and p^*_x is called the *reference probability* of x.



Utility

Axiom: continuity

If $a \succsim b \succsim c$ then there is some $p \in [0,1]$, such that:

$$b \sim [p : a|(1-p) : c]$$

Interpretation: Every intermediate prize is preferred equally to some lottery of the two extremal prizes.

Definition (Utility of a prize)

Define function $u : \Omega \to \mathbb{R}$, such that if $\omega \sim \ell^*_\omega = [p^*_\omega : \omega_M|(1-p^*_\omega) : \omega_m]$, then $u(\omega) = V_B(\ell^*_\omega)$ (where $0 \leqslant p^*_\omega \leqslant 1$).

Interpretation: The utility of a prize is proportional to the reference probability of the prize; specifically $u(\omega) = p^*_\omega(v(\omega_M) - v(\omega_m)) + v(\omega_m)$.

Preferences over lotteries

- Ultimately decisions must involve preference over lotteries/actions
- Define preference over lotteries, $\succsim_L$, as well as over outcomes

Definition (Lottery preference)

For lotteries ℓ and ℓ', we write $\ell \succsim_L \ell'$ iff ℓ is at least as preferred as ℓ'.

Definition (Inductive definition of lotteries)

For any $n \in \mathbb{N}$, and $p_1, \ldots, p_n$, where $0 \leqslant p_i \leqslant 1$ and $\sum_i p_i = 1$:

- if $\omega \in \Omega$ is a prize, then $[\omega]$ is a lottery
- if $\ell_1, \ldots, \ell_n$ are lotteries, then $[p_1 : \ell_1| \ldots |p_n : \ell_n]$ is a lottery

- Note that this means that lotteries in general may have other lotteries as prizes

Consistent preferences over lotteries

- What axioms characterise consistent preferences over lotteries?

Axiom L1: Monotonicity

If $A \succsim B$, then for the binary lotteries $[p : A|(1-p) : B]$ and $[p' : A|(1-p') : B]$:

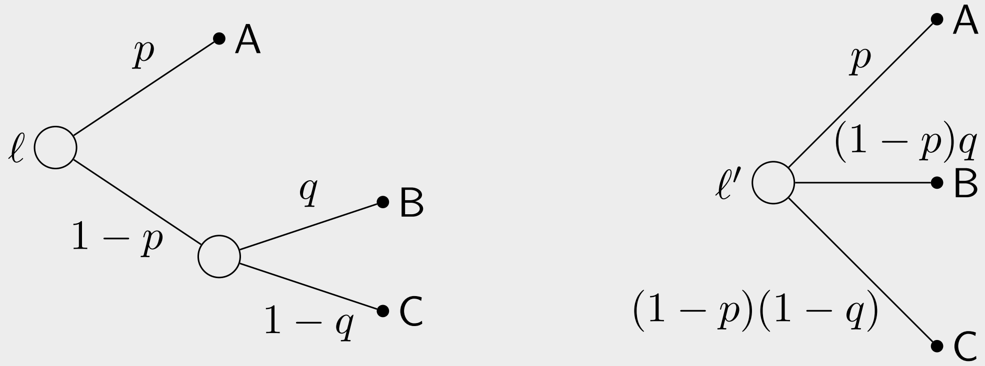
$$[p : A|(1-p) : B] \ \succsim_L \ [p' : A|(1-p') : B] \quad \text{iff} \quad p \geqslant p'.$$

- Interpretation: when the prizes in two lotteries are the same, the lottery which gives a better chance of the more preferred prize should be preferred
- This justifies the use of reference lotteries/probabilities to evaluate outcomes

Composite lotteries

Lotteries may have other lotteries as prizes; *i.e.*, they may be composed of other lotteries; *e.g.*,

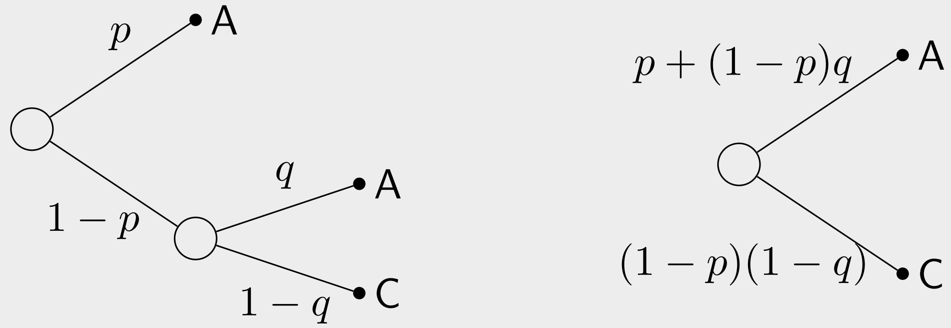
$$\ell = \big[p : A|1-p : [q : B|1-q : C]\big]$$



Agents should be indifferent between similar lotteries; *e.g.*, $\ell \sim_L \ell'$ above.

Composite lotteries: combination

Repeated outcomes can be combined/merged; *e.g.*,



These two should be equivalent:

$$\big[p : A | 1-p : [q : A | 1-q : C]\big] \sim_L [p + (1-p)q : A | (1-p)(1-q) : C]$$

Reduction of composite lotteries

Axiom: substitution of equivalents

If $\ell \sim \ell'$, then any substitution of one for the other in a composite lottery will yield lotteries that equally preferred.

Definition (Simple and composite lotteries)

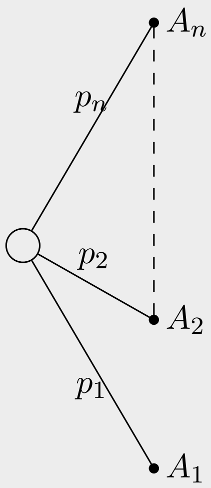
A *composite lottery* is one for which at least one prize is itself a lottery. A lottery which is not composite is said to be *simple*.

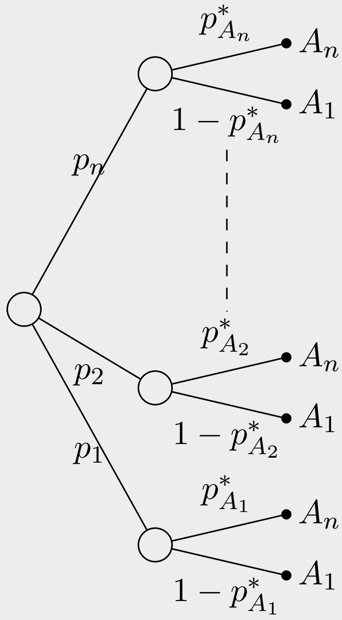
Axiom: lottery reduction

Composite lotteries can be reduced to equivalent (in regard to indifference) simple lotteries by combining probabilities in the usual way.

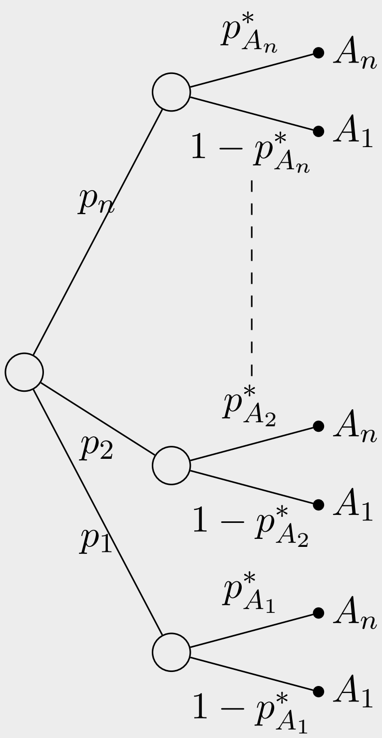
Normal lottery form

Suppose $A_n \succsim A_{n-1} \succsim \cdots \succsim A_1$, with $A_n \succ A_1$.
In lottery $\ell = [p_1 : A_1 | p_2 : A_2 | \ldots | p_n : A_n]$,
replace A_i with $[p^*_{A_i} : A_n | (1 - p^*_{A_i}) : A_1]$.

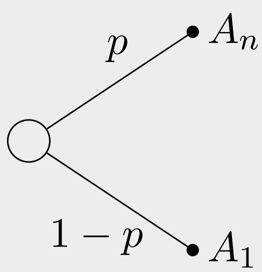




Standard lottery reduction



The lottery on the left can be combined to:



where
$p = p_1 p^*_{A_1} + p_2 p^*_{A_2} + \cdots + p_n p^*_{A_n}$.

Since $p^*_A = u(A)$, this gives:

$$p = p_1 u(A_1) + \cdots + p_n u(A_n)$$

Utility theory

Axioms

- *consistent preferences*: extended to lotteries
- *monotonicity*: between binary lotteries
- *substitution of equivalents*
- *reduction of composite lotteries*: by flattening, merging outcomes, and combining probabilities
- *continuity*: each outcome has an equivalent binary (standard) lottery

Theorem (Utility existence)

If the above axioms are satisfied, then there exists a linear function $u : \Omega \to \mathbb{R}$ such that $\omega_1 \succsim \omega_2$ iff $u(\omega_1) \geqslant u(\omega_2)$. Moreover, each u can be extended to a linear function U over lotteries, such that $\ell \succsim \ell'$ iff $U(\ell) \geqslant U(\ell')$, where $U(\ell) = V_B(\ell) = E(u)$.

The Maximal Utility Principle

Proof

By continuity assign $u(\omega) = p^*_\omega$ from ω's equivalent reference lottery ℓ^*_ω. Reduce each lottery ℓ to its equivalent reference lottery $[p_\ell : \omega_M|(1-p_\ell) : \omega_m]$. Moreover, by monotonicity $\ell \succsim \ell'$ iff $p_\ell \geqslant p_{\ell'}$; *i.e.*, iff $p_1 u(A_1) + \cdots + p_n u(A_n) \geqslant p'_1 u(A_1) + \cdots + p'_n u(A_n)$. But these are just $E_p(u) \geqslant E_{p'}(u)$. For lottery ℓ set:

$$U(\ell) = V_B(\ell) = E(u) = p_1 u(A_1) + \cdots + p_n u(A_n)$$

Maximal Utility Principle (MUP)

Rational agents prefer lotteries with greater expected utility over the prizes.

The MUP verifies that the *Bayes* decision rule applied to utilities is the rational rule to use in decision problems involving risk.

Utility: summary

- Preference is the fundamental notion in evaluating outcomes and actions/strategies
- Preference is a binary relation over outcomes/strategies/lotteries
- Consistent preferences lead to well-defined ‘utilities’ with which measure/quantify our preferences
- *Bayes* rule is *the* rational decision rule for evaluating strategies under risk