Utility theory

1 Utility theory
   - Preference relations
   - Ordering prizes/outcomes
   - Evaluating prizes
   - Evaluating lotteries
Utility theory

Outline

1 Utility theory
- Preference relations
- Ordering prizes/outcomes
- Evaluating prizes
- Evaluating lotteries

Evaluating outcomes and actions

Example (Bus or train?)

Would Alice prefer to catch the bus or the train if:
- she’s a doctor on an emergency call
- has an injured foot
- is a tourist.

- How to compare outcomes: travel time, walking distance, scenic appeal, comfort, etc.?
- How do we measure/quantify scenic appeal, comfort?
Preference and numbers

- So far preference based on numerical values assigned to outcomes and actions (i.e., on \( v \) and \( V \) respectively); i.e., an agent prefers:
  - outcome \( \omega_1 \) to \( \omega_2 \) if \( v(\omega_1) > v(\omega_2) \)
  - action \( A \) to \( B \) if \( V(A) > V(B) \)
- Does value (which?) determine preference or preference determine value?
- Can meaningful numbers always be assigned? e.g., Alice is a tourist who values comfort and good scenery
- Can rational decisions be made when numerical values aren’t given/available?

Numbers aren’t always required; consider the Maximin rule:

<table>
<thead>
<tr>
<th></th>
<th>( s_1 )</th>
<th>( s_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( v_{11} )</td>
<td>( v_{12} )</td>
</tr>
<tr>
<td>B</td>
<td>( v_{21} )</td>
<td>( v_{22} )</td>
</tr>
</tbody>
</table>


- \( s_1 \) \( s_2 \)
  | A | 20 | 0 |
  | B | 16 | 8 |

- \( s_1 \) \( s_2 \)
  | A | 9 | 2 |
  | B | 8 | 3 |

- \( Maximin \) is independent of specific values assigned to outcomes, provided preference order is preserved: i.e., \( v_{11} > v_{21} > v_{22} > v_{12} \)

**Exercise**

Will this be still be the case for Hurwicz’s rule (\( \alpha = \frac{1}{4} \))? miniMax Regret? Laplace’s rule?
Qualitative preference: preference without numbers

- *Maximin* can be reformulated in terms of *qualitative preferences* only

<table>
<thead>
<tr>
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<th>$s_1$</th>
<th>$s_2$</th>
<th>Preferences</th>
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<tbody>
<tr>
<td>A</td>
<td>$\omega_{11}$</td>
<td>$\omega_{12}$</td>
<td>$\omega_{11}$ preferred to $\omega_{21}$</td>
</tr>
<tr>
<td>B</td>
<td>$\omega_{21}$</td>
<td>$\omega_{22}$</td>
<td>$\omega_{21}$ preferred to $\omega_{22}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\omega_{22}$ preferred to $\omega_{12}$</td>
</tr>
</tbody>
</table>

**Definition (Qualitative *Maximin*)**

Associate an action with its/a least preferred outcome. Choose action whose associated outcome is most preferred.

- Which is least preferred outcome of A? *i.e.*, $\omega_{11}$ preferred to $\omega_{12}$?

Preference and value

Consequences of assigning numerical quantities (*i.e.*, via some value function $v : \Omega \rightarrow \mathbb{R}$) to encode preference:

- agent either prefers $a$ to $b$, or $b$ to $a$, or agent prefers them equally—agent *indifferent* between $a$ and $b$
- if agent prefers $a$ to $b$, and $b$ to $c$, then agent prefers $a$ to $c$; *i.e.*, preferences *transitive*

**Questions**

- Are these conditions justified in practice?
- Do actual (human) agents always behave in this way?
- Can you find counter-examples?
Rational decisions can be made without numerical values so long as an agent’s preferences are ‘consistent’.

What does ‘preference consistency’ mean?

Then, for example:

- $\omega_{11}$ preferred to $\omega_{12}$
- $\omega_{21}$ not preferred to $\omega_{11}$

A rational agent’s (strict) preferences should be consistent in the sense that, e.g., an agent that:

- prefers apples (A) to bananas (B) shouldn’t prefer bananas to apples
- prefers apples (A) to bananas (B) and bananas (B) to carrots (C) shouldn’t prefer carrots (C) to apples (A)

Exercises

- What would be consequences of the failure of the first property above?
- In the second property above, should the agent then necessarily prefer apples to carrots?

Preferences is a binary relation.
Binary relations: overview

Modelling binary relations:

- If $A$ and $B$ are sets, define the *Cartesian product* of $A$ and $B$: $A \times B = \{(a, b) \mid a \in A \land b \in B\}$; e.g., the set of all coordinate pairs on the Euclidean plane $\mathbb{R} \times \mathbb{R}$

**Definition (Binary relation)**

A *binary relation* $R$ from $A$ to $B$ is a subset of $A \times B$; i.e., $R \subseteq A \times B$. Each ordered pair $(x, y) \in R$ is called an *instance* of $R$.

- In *infix notation*: $aRb$ iff $(a, b) \in R$; e.g., $3 \leq 5$
- If $aRb$ (i.e., $(x, y) \in R$) then the relation $R$ is said to *hold* for $x$ with $y$; e.g., because $3 \leq 5$, then $\leq$ holds for $3$ with $5$

**Definition (Binary relation on a set $A$)**

A binary relation, $R$, on a set $A$ is a subset of $A \times A$; i.e., $R \subseteq A \times A$.

- e.g., the binary relation ‘is greater than’, written $> \subseteq \mathbb{R} \times \mathbb{R}$, is a binary relation on the set of real numbers $\mathbb{R}$ (and on $\mathbb{N}$, and on $\mathbb{Q}$)
- e.g., the ‘greater than’ relation ($>$) holds between real numbers $3$ and $\pi$ (written $3 > \pi$); i.e., $3 > \pi$ is an instance of $>$
Representing relations

- Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$, then a relation $R \subseteq A \times B$ can be represented by the matrix/table:

```
R | b_1   b_2   b_3   b_4
---|--------|--------|--------|--------|
a_1|x      x      |        |
a_2|x      x      | x      |
a_3|x      |        |        |
```

- An $\times$ appears at entry in row $x$ and column $y$ iff $xRy$. More succinctly, there’s a $\times$ at $(x,y)$ iff $xRy$ iff $(x,y)$ is an instance of $R$. e.g., above $a_1Rb_2$, $a_2Rb_1$, and $a_3Rb_4$, but $a_1Rb_1$.

Relational properties

Let $R$ be a binary relation on some set $A$:

- $R$ is reflexive iff for every $x \in A$, $xRx$; e.g., for every $x \in \mathbb{R}$, $x = x$, $x \leq x$, $x \geq x$
- $R$ is irreflexive iff for every $x \in A$, $xRx$ does not hold; e.g., for every $x \in \mathbb{R}$, $x \neq x$, $x < x$, $x > x$ do not hold
- $R$ is transitive iff for any $x, y, z \in A$, when $xRy$ and $yRz$, then $xRz$; e.g., $=, <, \leq$ on $\mathbb{R}$
- $R$ is symmetric iff for any $x, y \in A$, when $xRy$, then $yRx$; e.g., $=, \leq$ on $\mathbb{R}$
- $R$ is total iff $xRy$ or $yRx$; e.g., $=, \leq$ on $\mathbb{R}$
- $R$ is asymmetric iff whenever $xRy$ then $yRx$ does not hold; e.g., $<$ on $\mathbb{R}$
- $R$ is antisymmetric iff whenever $xRy$ and $yRx$, then $x = y$; e.g., $\leq$ on $\mathbb{R}$
Preference relations

- A binary relation can be used to model *strict preference*:

**Definition**

An agent *strictly prefers* element \( a \) to \( b \), written \( a \succ b \), iff it prefers \( a \) more than \( b \); i.e., it would eliminate \( b \). The collection of all such instances comprises the agent’s *strict preference relation*, \( \succ \).

- What intuitive properties should strict preference relations satisfy?
  - if \( x \succ y \), then it should not be the case that \( y \succ x \)
  - if \( x \succ y \) and \( y \succ z \), then it should not be the case that \( z \succ x \)

Representing preference: Hasse diagrams

- Suppose we’re given the following strict preferences on a set \( A = \{a, b, c, d, e, f, g\} \):

\[
\begin{align*}
    a & \succ g & f & \succ c & d & \succ a & a & \succ e \\
    e & \succ b & c & \succ a & g & \succ b \\
\end{align*}
\]

- Do we *know* that \( a \succ b \)? What about \( c \succ d \)? \( f \succ d \)? \( e \succ g \)?
- \( x \succ y \) iff there’s a path following arrows from \( x \) to \( y \)
Indifference: equal preference

**Definition (Indifference)**

If two elements \( a \) and \( b \) are *equally preferred* then the agent is said to be *indifferent* between them, written \( a \sim b \). The set of all such instances constitutes an agent’s binary relation of *indifference*. The *indifference class* of \( a \) is \( [a] = \{ b \mid a \sim b \} \).

**Definition (Weak preference)**

Element \( a \) is *weakly preferred* to \( b \), written \( a \preccurlyeq b \), iff \( a \) is strictly preferred to \( b \) or the two are equally preferred; i.e., \( a \) is at least as preferred as \( b \); i.e., \( a \preccurlyeq b \) iff \( a \succ b \) or \( a \sim b \).

Representing preference: Hasse diagrams

- Suppose we’re given the following strict preferences on a set \( A = \{a, b, c, d, e, f, g\} \):

  \[
  \begin{align*}
  a & \succ g \\
  f & \succ c \\
  d & \succ a \\
  a & \succ e \\
  e & \succ b \\
  c & \succ a \\
  g & \succ b 
  \end{align*}
  \]

- Do we *know* that \( a \succ b \)? What about \( c \succ d \)? \( f \succ d \)? \( e \succ g \)?
- \( x \succ y \) iff there’s a path following arrows from \( x \) to \( y \)
Indifference properties

The following are intuitive properties of indifference:

- if $x \sim y$, then $y \sim x$
- if $x \sim y$ and $y \sim z$, then $x \sim z$
- $x \sim x$ holds for any $x \in A$

Combined properties:

- if $x \sim y$ and $z \succ x$, then $z \succ y$
- if $x \sim y$ and $x \succ z$, then $y \succ z$

Note that, in the previous problem, it would be inconsistent for $c \sim d$ and $f \sim d$, as $f \succ c$, which would imply $f \succ d$.

Axiomatisation of consistent weak preference

- What does it mean for an agent’s preferences to be consistent/rational?
- Regard $\succeq$ as the fundamental/primitive notion, and interpret $x \succeq y$ as “$x$ is at least as preferred as $y$”
- The following axioms characterise consistent preference

**Axiom 1: Transitivity**

The relation $\succeq$ is transitive; i.e., preference accumulates.

**Axiom 2: Comparability**

The relation $\succeq$ is total; i.e., every outcome is comparable.
## Derived definitions

From the basic definition of $\succeq$ we can define indifference and strict preference as derived notions:

**Definition (Indifference)**

The relation of *indifference*, denoted $\sim$, is defined by:

$$x \sim y \text{ iff } x \succeq y \text{ & } y \succeq x.$$

**Definition (Strict preference)**

The relation of *strict preference*, denoted $\succ$, is defined by:

$$x \succ y \text{ iff } y \succeq x \text{ does not hold.}$$

## Ordinal value functions

**Definition (Ordinal value function)**

An *ordinal value function* on a ‘preference set’ $(A, \succeq)$ is a function $v : A \rightarrow \mathbb{R}$ such that $v(x) \geq v(y)$ iff $x \succeq y$.

**Exercise**

Show that for any ordinal value function $v$:

- $v(x) > v(y)$ iff $x \succ y$
- $v(x) = v(y)$ iff $x \sim y$

**Theorem (Consistency)**

*For any consistent preference relation there exists an ordinal value function.*
Preference relations

Properties

The following properties follow from the earlier definitions:
- If an agent’s preferences are consistent then ~ is an equivalence relation.
- The corresponding strict preference relation ≻ is a strict total order.
- Strict preference satisfies an indifference version of the trichotomy law i.e., exactly one of the following holds between any $x, y \in A$: $x \succ y$ or $x \sim y$ or $y \succ x$.

Weak preference

Consider the following complete listing of weak preferences on a set $A = \{a, b, c, d\}$:

\[
\begin{align*}
    a \preceq c & \quad c \preceq a & \quad b \succeq d & \quad d \preceq a & \quad d \preceq c \\
    a \preceq a & \quad c \preceq c & \quad b \preceq b & \quad d \preceq d \\
    b \succeq a & \quad b \preceq c \\
    a \preceq c & \quad c \preceq a & \quad b \succeq d & \quad d \preceq a & \quad d \preceq c \\
    a \sim c & \quad b \succ d & \quad d \succ a & \quad d \succ c \\
\end{align*}
\]

\[
\begin{array}{c}
    b \\
    \downarrow \quad d \\
    \quad \downarrow \quad a \\
    \quad \downarrow c
\end{array}
\]

\[
\begin{array}{c}
    b \\
    \quad \rightarrow d \\
    \quad \rightarrow a \\
    \quad \rightarrow c
\end{array}
\]
Generating rankings

<table>
<thead>
<tr>
<th>≻</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
</tr>
<tr>
<td>c</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d</td>
<td></td>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>≻_I</th>
<th>{b}</th>
<th>{d}</th>
<th>{a, c}</th>
</tr>
</thead>
<tbody>
<tr>
<td>{b}</td>
<td>×</td>
<td>×</td>
<td></td>
</tr>
<tr>
<td>{d}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{a, c}</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- The rank of $x$ is $r(x) =$ number of $\times$ in $x$’s row; e.g.,
  $r(b) = 2$, $r(d) = 1$, and $r(a) = r(c) = 0$.
- This ranking is an ordinal value function.

Ranking

Definition (Rank)

The rank of an indifference is defined by the successive values assigned to indifference class when the lowest indifference class is assigned rank 0.

i.e., the ranks above are 0, 1, 2, …
Evaluating intermediate prizes

- Suppose the prizes in a lottery $\ell$ have been ordered by preference: $a \succ b \succ c \succ d$.
- Choose fixed reference values for the best and worst prizes, $a$ and $d$: e.g., $v(a) = 100$ and $v(d) = 0$

\[
\begin{array}{cccc}
0 & d & c & b & 100 \\
\end{array}
\]

\[
\begin{array}{cccc}
\end{array}
\]

\[
\begin{array}{cccc}
\end{array}
\]

- Which values should be assigned to $b$? $100 \times \text{rank}(b)/\text{rank}(a)$?
- Agent’s preferences: $b \sim [\frac{3}{4} : a]_{\frac{1}{4}} : d$
- Then $v(b)$ should be $V_B([\frac{3}{4} : a]_{\frac{1}{4}} : d)$; i.e., $v(b) = \frac{3}{4} \times 100 = 75$

\[
\begin{array}{cccc}
\end{array}
\]

\[
\begin{array}{cccc}
\end{array}
\]

In general, for prize $x$ such that $x \sim [p_x : a](1 - p_x) : d$, for $0 \leq p_x \leq 1$, assign value $v(x)$, where:

\[
\frac{v(x) - v(d)}{v(a) - v(d)} = p_x
\]

i.e., $v(x) = \alpha p_x + \beta$, where $\alpha = v(a) - v(d)$ and $\beta = v(d)$
Binary lotteries

Definition (Binary lottery)

A *binary lottery* is a lottery in which at most two possible prizes have non-zero probability: *i.e.*, of the form \( \ell = [p : A \mid (1 - p) : B] \).

\[ p \rightarrow A \]

\[ 1 - p \rightarrow B \]

e.g., the lottery for tossing a fair coin: \( \ell = [\frac{1}{2} : h \mid \frac{1}{2} : t] \).

Reference lotteries

Definition (Reference lottery)

Let \( \omega_M \) and \( \omega_m \) be, respectively, the best and worst possible prizes \((\omega_M \succ \omega_m)\). A *reference lottery*, \( \ell^* \), is a binary lottery:

\[ \ell^* = [p : \omega_M \mid (1 - p) : \omega_m] \]

If prize \( x \sim \ell^*_x = [p^*_x : \omega_M \mid (1 - p^*_x) : \omega_m] \), then \( \ell^*_x \) is called the *reference lottery* for \( x \), and \( p^*_x \) is called the *reference probability* of \( x \).
Utility

**Axiom: continuity**

If \( a \succeq b \succeq c \) then there is some \( p \in [0, 1] \), such that:

\[
\text{ } b \sim [p : a : (1 - p) : c] \text{.}
\]

Interpretation: Every intermediate prize is preferred equally to some lottery of the two extremal prizes.

**Definition (Utility of a prize)**

Define function \( u : \Omega \to \mathbb{R} \), such that if \( \omega \sim \ell^* = [p^*_\omega : \omega_M | (1 - p^*_\omega) : \omega_m] \), then \( u(\omega) = V_B(\ell^*_\omega) \) (where \( 0 \leq p^*_\omega \leq 1 \)).

Interpretation: The utility of a prize is proportional to the reference probability of the prize; specifically \( u(\omega) = p^*_\omega (v(\omega_M) - v(\omega_m)) + v(\omega_m) \).

Preferences over lotteries

- Ultimately decisions must involve preference over lotteries/actions
- Define preference over lotteries, \( \succeq_L \), as well as over outcomes

**Definition (Lottery preference)**

For lotteries \( \ell \) and \( \ell' \), we write \( \ell \succeq_L \ell' \) iff \( \ell \) is at least as preferred as \( \ell' \).

**Definition (Inductive definition of lotteries)**

For any \( n \in \mathbb{N} \), and \( p_1, \ldots, p_n \), where \( 0 \leq p_i \leq 1 \) and \( \sum_i p_i = 1 \):

- if \( \omega \in \Omega \) is a prize, then \( [\omega] \) is a lottery
- if \( \ell_1, \ldots, \ell_n \) are lotteries, then \( [p_1 : \ell_1 | \ldots | p_n : \ell_n] \) is a lottery

- Note that this means that lotteries in general may have other lotteries as prizes
Consistent preferences over lotteries

- What axioms characterise consistent preferences over lotteries?

**Axiom L1: Monotonicity**

If \( A \succeq B \), then for the binary lotteries \([p : A|(1-p) : B]\) and \([p' : A|(1-p') : B]\):

\[
[p : A|(1-p) : B] \succeq_L [p' : A|(1-p') : B] \text{ iff } p \geq p'.
\]

- Interpretation: when the prizes in two lotteries are the same, the lottery which gives a better chance of the more preferred prize should be preferred
- This justifies the use of reference lotteries/probabilities to evaluate outcomes

Composite lotteries

Lotteries may have other lotteries as prizes; *i.e.*, they may be composed of other lotteries; *e.g.*,\

\[
\ell = [p : A|1-p : [q : B|1-q : C]]
\]

Agents should be indifferent between similar lotteries; *e.g.*, \( \ell \sim_L \ell' \) above.
Composite lotteries: combination

Repeated outcomes can be combined/merged; \( e.g., \)

\[
\begin{array}{c}
p \cdot A \\
1 - p \\
q \cdot A \\
1 - q \cdot C
\end{array}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\begin{array}{c}
p + (1 - p)q \cdot A \\
(1 - p)(1 - q) \cdot C
\end{array}
\]

These two should be equivalent:

\[
\]

Reduction of composite lotteries

**Axiom: substitution of equivalents**

If \( \ell \sim \ell' \), then any substitution of one for the other in a composite lottery will yield lotteries that equally preferred.

**Definition (Simple and composite lotteries)**

A composite lottery is one for which at least one prize is itself a lottery. A lottery which is not composite is said to be simple.

**Axiom: lottery reduction**

Composite lotteries can be reduced to equivalent (in regard to indifference) simple lotteries by combining probabilities in the usual way.
Normal lottery form

Suppose $A_n \succsim A_{n-1} \succsim \cdots \succsim A_1$, with $A_n \succ A_1$.

In lottery $\ell = [p_1 : A_1 | p_2 : A_2 | \ldots | p_n : A_n]$, replace $A_i$ with $[p^*_A : A_n | (1 - p^*_A) : A_1]$.

The lottery on the left can be combined to:

\[
p = p_1 p^*_A + p_2 p^*_A + \ldots + p_n p^*_A.
\]

where $p^*_A = u(A)$, this gives:

\[
p = p_1 u(A_1) + \cdots + p_n u(A_n)
\]
Utility theory

Axioms

- **consistent preferences**: extended to lotteries
- **monotonicity**: between binary lotteries
- **substitution of equivalents**
- **reduction of composite lotteries**: by flattening, merging outcomes, and combining probabilities
- **continuity**: each outcome has an equivalent binary (standard) lottery

**Theorem (Utility existence)**

If the above axioms are satisfied, then there exists a linear function $u : \Omega \rightarrow \mathbb{R}$ such that $\omega_1 \succeq \omega_2$ iff $u(\omega_1) \geq u(\omega_2)$. Moreover, each $u$ can be extended to a linear function $U$ over lotteries, such that $\ell \succeq \ell'$ iff $U(\ell) \geq U(\ell')$, where $U(\ell) = V_B(\ell) = E(u)$.

**The Maximal Utility Principle**

**Proof**

By continuity assign $u(\omega) = p^*_\omega$ from $\omega$’s equivalent reference lottery $\ell^*_\omega$. Reduce each lottery $\ell$ to its equivalent reference lottery $[p_\ell : \omega_M](1 - p_\ell) : \omega_m]$. Moreover, by monotonicity $\ell \succeq \ell'$ iff $p_\ell \geq p_{\ell'}$; i.e., iff $p_1u(A_1) + \cdots + p_nu(A_n) \geq p'_1u(A_1) + \cdots + p'_nu(A_n)$. But these are just $E_p(u) \geq E_{p'}(u)$. For lottery $\ell$ set:

$$U(\ell) = V_B(\ell) = E(u) = p_1u(A_1) + \cdots + p_nu(A_n)$$

**Maximal Utility Principle (MUP)**

Rational agents prefer lotteries with greater expected utility over the prizes.

The MUP verifies that the Bayes decision rule applied to utilities is the rational rule to use in decision problems involving risk.
Utility: summary

- Preference is the fundamental notion in evaluating outcomes and actions/strategies
- Preference is a binary relation over outcomes/strategies/lotteries
- Consistent preferences lead to well-defined ‘utilities’ with which measure/quantify our preferences
- Bayes rule is the rational decision rule for evaluating strategies under risk