Exercise 1. Show that the algorithm solving \textsc{Comp-FVS} from the lecture notes has running time $O^*(4^k)$.

\textbf{Hint.} Use the measure $k + \text{cc}(S)$, where \text{cc}(S) is the number of connected components of $G[S]$.

Exercise 2. Recall that a \textit{cluster graph} is a graph where every connected component is a complete graph.

\begin{tabular}{|l|}
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\textsc{Cluster Vertex Deletion} \\
\hline
\textbf{Input:} & Graph $G = (V,E)$, integer $k$ \\
\textbf{Parameter:} & $k$ \\
\textbf{Question:} & Is there a set of vertices $S \subseteq V$ with $|S| \leq k$ such that $G - S$ is a cluster graph? \\
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\end{tabular}

Recall that $G$ is a cluster graph iff $G$ contains no induced $P_3$.

- Design an $O^*(2^k)$ time algorithm for \textsc{Cluster Vertex Deletion}.

\textbf{Hints.} (1) Show that the disjoint version of the problem can be solved in polynomial time: given $(G = (V,E), S, k)$ such that $|S| = k+1$ and $G - S$ is a cluster graph, find a $S^* \subseteq V \setminus S$ with $|S^*| \leq k$ such that $G - S^*$ is a cluster graph.

(2) Simplification rule for $v \in V \setminus S$ inducing a $P_3$ with 2 vertices in $S$. Reduce to maximum weight matching.

\textbf{Solution sketch.}

\begin{tabular}{|l|}
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\textsc{Disjoint-CVD} \\
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\textbf{Input:} & graph $G = (V,E)$, integer $k$, cluster vertex deletion set $S$ of size $k + 1$ of $G$ \\
\textbf{Output:} & a cluster vertex deletion set $S^*$ of $G$ with $|S^*| \leq k$ and $S^* \cap S = \emptyset$, if one exists \\
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\end{tabular}

Simplification rules:

- If $G[S]$ contains an induced $P_3$, then return \textbf{No}.

- If $\exists v \in V \setminus S$ such that $G[S \cup \{v\}]$ contains an induced $P_3$, then set $G \leftarrow G - v$ and $k \leftarrow k - 1$.

Now each vertex in $V \setminus S$ has either no neighbor in $S$ or is adjacent to all the vertices of exactly one cluster of $G[S]$. Reduce the problem to maximum weighted matching in a bipartite graph where one independent set corresponds to the clusters in $G[S]$ and each vertex in the other independent set corresponds to cliques neighboring exactly one cluster in $G[S]$. It remains to define the edges of the auxiliary graph and their weights.