Reasoning about (Lack of) Knowledge

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COMP4418, Week 7
Motivation

John McCarthy (1927–2011):
- Stanford, MIT, Dartmouth
- Turing Award
- Invented Lisp (1958)
- Invented Garbage Collection (1959)
- Founding Father of AI (with Minsky, Newell, Simon, 1955)
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  - Programs with Common Sense
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    - Imperative conclusion: take action
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Advice Taker motivates (directly or indirectly) a lot of AI research, in particular what we’ll be studying for the next three weeks
Observation: Non-knowledge is important

Not only what we know is relevant, but also what we don’t know
Motivation

Observation: Non-knowledge is important

Not only what we know is relevant, but also what we *don’t* know

You don’t know what’s in the gift box.
You’ll treat it with great care.
Motivation

Observation: Non-knowledge is important

Not only what we know is relevant, but also what we *don’t* know

You know Jane has a phone, but you don’t know her number.

You’ll look it up.
Motivation

Observation: Non-knowledge is important

Not only what we know is relevant, but also what we *don’t* know

You know Jane is holding ace of spades *or* of hearts, but not which.
You’ll need a strategy that wins in either case.
Motivation

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Not only what we know is relevant, but also what we don’t know.

You know Jane is holding ace of spades or of hearts, but not which. You’ll need a strategy that wins in either case.

How can we accurately capture knowledge and non-knowledge?
Overview of the Lecture

- A Logic of Knowledge – The Propositional Fragment
  - Why not classical logic?
  - Syntax and semantics
  - Omniscience, introspection, only-knowing
  - Representation theorem

- A Logic of Knowledge – The First-Order Case

- Extensions of the Logic of Knowledge
What Is a Knowledge Base?

- A **knowledge base** (KB) is a collection of sentences that describe (a fragment of) the world.
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- KB completely characterises what the agent knows, i.e.,
  - $\alpha$ is known $\implies$ $\text{KB} \models \alpha$
  - $\alpha$ is not known $\implies$ $\text{KB} \not\models \alpha$

  $\implies$ KB is *all* the agent knows.
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\[ \implies \text{KB is all the agent knows} \]

- Purpose: evaluate queries
  - What is known? What is unknown?
  - Similar to a database, but draws interences
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Purpose: evaluate queries

- What is known? What is unknown?
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Usually: what is known \( \subset \) what is true

- Agent’s knowledge is incomplete
- Agent should be aware of that
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  - What is known? What is unknown?
  - Similar to a database, but draws inferences.

- Usually: what is known \( \subsetneq \) what is true
  - Agent’s knowledge is incomplete
  - Agent should be aware of that.

- Usually: knowing is more than database lookup
  - \( \alpha \in KB \implies \alpha \) is explicit knowledge (= database lookup)
  - \( KB \models \alpha \implies \alpha \) is implicit knowledge (= logical inference)
  - Usually: explicit knowledge \( \subsetneq \) (implicit) knowledge.
Why Not Classical Logic?

Suppose all you know is \((p \lor q \lor r) \land (p \lor q \lor \neg r)\)

Then:

1. You don’t know that \(r\).
2. You don’t know that \(\neg r\).
Why Not Classical Logic?

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3. You know that \(p\) or \(q\).
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4. You don’t know that \(p\).
5. You don’t know that \(q\).
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4. You don’t know that \(p\).
5. You don’t know that \(q\).
6. You know that \(p\) or \(q\), but not which.
Why Not Classical Logic?

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Then:

1. You don’t know that \(r\). \(\text{KB} \not \models r\)
2. You don’t know that \(\neg r\). \(\text{KB} \not \models \neg r\)
3. You know that \(p\) or \(q\).
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2. You don’t know that \(\neg r\). \(\text{KB} \nmid \neg r\)
3. You know that \(p\) or \(q\). \(\text{KB} \models (p \lor q)\)
4. You don’t know that \(p\).
5. You don’t know that \(q\).
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Then:
1. You don’t know that \(r\). \(\text{KB} \nvDash r\)
2. You don’t know that \(\neg r\). \(\text{KB} \nvDash \neg r\)
3. You know that \(p\) or \(q\). \(\text{KB} \models (p \lor q)\)
4. You don’t know that \(p\). \(\text{KB} \nvDash p\)
5. You don’t know that \(q\). \(\text{KB} \nvDash q\)
6. You know that \(p\) or \(q\), but not which. \(\text{KB} \models ???\)

**Problem:** Classical logic cannot express 6 directly in one formula
Why Not Classical Logic?

Suppose all you know is \((p \lor q \lor r) \land (p \lor q \lor \neg r) \land \neg k_p \land \ldots\)

Then:

1. You don’t know that \(r\). \quad \text{KB} \models \neg k_r
2. You don’t know that \(\neg r\). \quad \text{KB} \models \neg k_{\neg r}
3. You know that \(p\) or \(q\). \quad \text{KB} \models k_{p\lor q}
4. You don’t know that \(p\). \quad \text{KB} \models \neg k_p
5. You don’t know that \(q\). \quad \text{KB} \models \neg k_q
6. You know that \(p\) or \(q\), but not which.
   \[\text{KB} \models k_{p\lor q} \land \neg k_p \land \neg k_p\]

Problem: Classical logic cannot express 6 directly in one formula.

Idea #1: Compile \((p \lor q \lor c)\) to new atoms \(k_p, k_{p\lor q}, \ldots\) \(\times\)

Does not scale.
Why Not Classical Logic?

Suppose all you know is \((p \lor q \lor r) \land (p \lor q \lor \lnot r)\)

Then:

1. You don’t know that \(r\). \(\text{KB} \models r = \text{U}\)
2. You don’t know that \(\lnot r\). \(\text{KB} \models \lnot r = \text{U}\)
3. You know that \(p\) or \(q\). \(\text{KB} \models (p \lor q)\)
4. You don’t know that \(p\). \(\text{KB} \models p = \text{U}\)
5. You don’t know that \(q\). \(\text{KB} \models q = \text{U}\)
6. You know that \(p\) or \(q\), but not which.
   \(\text{KB} \models (p \lor q) \land p = \text{U} \land q = \text{U}\)

**Problem:** Classical logic cannot express 6 directly in one formula

Idea #2: Three-valued logic \(\{0, 1, \text{U}\}\)  

How would \(\text{U} \lor \text{U}\) behave? Is it known? Unknown?
Why Not Classical Logic?

Suppose all you know is $(p \lor q \lor r) \land (p \lor q \lor \neg r)$

Then:

1. You don’t know that $r$.  \[ \text{OKB} \models \neg \text{K}r \]
2. You don’t know that $\neg r$.  \[ \text{OKB} \models \neg \text{K} \neg r \]
3. You know that $p$ or $q$. \[ \text{OKB} \models \text{K}(p \lor q) \]
4. You don’t know that $p$. \[ \text{OKB} \models \neg \text{K}p \]
5. You don’t know that $q$. \[ \text{OKB} \models \neg \text{K}q \]
6. You know that $p$ or $q$, but not which. \[ \text{OKB} \models \text{K}(p \lor q) \land \neg \text{K}p \land \neg \text{K}q \]

**Problem:** Classical logic cannot express 6 directly in one formula

**Idea #3:** Add unary operators $\mathcal{O}$ and $\text{K}$ to express knowledge  ✓
The Language of $\mathcal{OL}_{PL}$

The language of only-knowing (propositional fragment) $\mathcal{OL}_{PL}$:

- $p, q, r, \ldots$ \hspace{1cm} atomic propositions
- $\neg \alpha$ \hspace{1cm} “not $\alpha$”
- $(\alpha \lor \beta)$ \hspace{1cm} “$\alpha$ or $\beta$”
- $(\alpha \land \beta)$ \hspace{1cm} “$\alpha$ and $\beta$”
- $(\alpha \rightarrow \beta)$ \hspace{1cm} “$\alpha$ implies $\beta$”
- $(\alpha \leftrightarrow \beta)$ \hspace{1cm} “$\alpha$ is equivalent to $\beta$”
- $K\alpha$ \hspace{1cm} “$\alpha$ is known”
- $O\alpha$ \hspace{1cm} “$\alpha$ is all that is known”
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  “$\alpha$ is known”
- $O\alpha$  
  “$\alpha$ is all that is known”
Recap: Technical Terms (1)

A logical language is a *formal language* over an *alphabet* (here: \{p, q, r, \ldots, (, ), \neg, \lor, K, O\}) and a *grammar* (previous slide), i.e., rules that allow us to phrase sentences in that language.
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The sentences carry *no meaning by themselves*. We define a *model theory* to give them a *semantics*, i.e., to define what sort of formal structure interprets a sentence. Such an *interpretation I* satisfies a sentence \( \alpha \), written \( I \models \alpha \), or falsifies it, written \( I \not\models \alpha \).
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A typical rule of a semantics is

\[
I \models (\alpha \lor \beta) \text{ if and only if } I \models \alpha \text{ or } I \models \beta.
\]

Note that \( \lor \) is a symbol of the logical language, whereas “if and only if” and “or” are natural language expressions. The rule says that the symbol “\( \lor \)” corresponds to the natural language expression “or”.
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Note that \(\lor\) is a symbol of the logical language, whereas “if and only if” and “or” are natural language expressions. The rule says that the symbol “\(\lor\)” corresponds to the natural language expression “or”.

We will sometimes take the liberty to omit brackets to ease readability. For instance, we write \((p \lor q \lor r)\) instead of \(((p \lor q) \lor r)\) or \((p \lor (q \lor r))\), implicitly assuming our semantics of \(\lor\) is associative.
Recap: Technical Terms (2)

The form of such an interpretation varies between logics. Propositional logic uses truth tables, first-order logic usually uses structures with a domain and interpretation function.

- When an interpretation $I$ satisfies a sentence, we write $I \models \alpha$.
- When all interpretations satisfy a sentence $\alpha$, then $\alpha$ is valid and we write $\models \alpha$.
- When all interpretations that satisfy some sentence $\Sigma$ or set of sentences $\Sigma$ also satisfy $\alpha$, we say $\Sigma$ entails $\alpha$ and write $\Sigma \models \alpha$. 

Different semantics are possible. Typically there is a proof theory, and model theory and proof theory should be equivalent ($\models \alpha$ if and only if $\vdash \alpha$). Nevertheless, we will only focus on the semantics in the next weeks.
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Different semantics are possible. What justifies a semantics? Typically there is a proof theory, and model theory and proof theory should be equivalent ($\models \alpha$ if and only if $\vdash \alpha$). Nevertheless, we will only focus on the semantics in the next weeks.
The Semantics of $\mathcal{OL}_{PL}$

**Definition: semantics of $\mathcal{OL}_{PL}$**

A **world** $w$ is a function from the atomic propositions to $\{0, 1\}$. 
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Definition: semantics of $\mathcal{OL}_{PL}$

A **world** $w$ is a function from the atomic propositions to $\{0, 1\}$.

- $w \models P \iff w[P] = 1$
- $w \models \neg \alpha \iff w \not\models \alpha$
- $w \models (\alpha \lor \beta) \iff w \models \alpha \text{ or } w \models \beta$
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- $w \models K\alpha \iff \text{???}$
- $w \models O\alpha \iff \text{???}$
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An epistemic state $e$ is a set of worlds.

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- $e, w \models \neg \alpha \iff e, w \not\models \alpha$
- $e, w \models (\alpha \lor \beta) \iff e, w \models \alpha$ or $e, w \models \beta$
- $e, w \models K\alpha \iff ???$
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- $e, w \models (\alpha \lor \beta) \iff e, w \models \alpha$ or $e, w \models \beta$
- $e, w \models K\alpha \iff$ for all worlds $w', w' \in e \Rightarrow e, w' \models \alpha$
- $e, w \models O\alpha \iff$ ???

“$\Rightarrow$” stands for natural language expressions “only if”.
“$\iff$” and “$\iff$” stand for natural language expressions “if and only if”.
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- $e, w \models K\alpha \iff \text{for all worlds } w', \ w' \in e \Rightarrow e, w' \models \alpha$
- $e, w \models O\alpha \iff \text{for all worlds } w', \ w' \in e \iff e, w' \models \alpha$

“$\Rightarrow$” stands for natural language expressions “only if”.
“$\iff$” and “$\iff$” stand for natural language expressions “if and only if”.


Abbreviations

Recall:

- \( (\alpha \land \beta) \overset{\text{def}}{=} \neg(\neg \alpha \lor \neg \beta) \)
- \( (\alpha \rightarrow \beta) \overset{\text{def}}{=} (\neg \alpha \lor \beta) \)
- \( (\alpha \leftrightarrow \beta) \overset{\text{def}}{=} (\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha) \)

\(\land\) should be “and”
\(\rightarrow\) should be “only if”
\(\leftrightarrow\) should be “if and only if”.

Lemma: abbreviations

\[ e, w \models \alpha \land \beta \iff e, w \models \alpha \text{ and } e, w \models \beta \]
\[ e, w \models \alpha \rightarrow \beta \iff e, w \models \alpha \implies e, w \models \beta \]
\[ e, w \models \alpha \leftrightarrow \beta \iff e, w \models \alpha \iff e, w \models \beta \]

Proof on paper
Some Lemmas

**Definition: objective, subjective**

If \( \phi \) mentions no atoms inside \( K \) or \( O \), we say \( \phi \) is **objective**. If \( \sigma \) mentions atoms only inside \( K \) or \( O \), we say \( \sigma \) is **subjective**.

- \( ((p \lor q) \land p \land q) \) is objective
- \( K((p \lor q) \land \neg Kp \land \neg Kq) \) is subjective
Some Lemmas

**Definition: objective, subjective**

If $\phi$ mentions no atoms inside $K$ or $O$, we say $\phi$ is **objective**. If $\sigma$ mentions atoms only inside $K$ or $O$, we say $\sigma$ is **subjective**.

- $(p \lor q) \land p \land q$ is objective
- $K((p \lor q) \land \neg Kp \land \neg Kq)$ is subjective

**Lemma: objective, subjective**

Let $\phi$ be objective. Then $e, w \models \phi \iff e', w \models \phi$.

Let $\sigma$ be subjective. Then $e, w \models \sigma \iff e, w' \models \sigma$. 
Some Lemmas

**Definition: objective, subjective**

If \( \phi \) mentions no atoms inside \( K \) or \( O \), we say \( \phi \) is **objective**.  
If \( \sigma \) mentions atoms only inside \( K \) or \( O \), we say \( \sigma \) is **subjective**.

- \([(p \lor q) \land p \land q]\) is objective
- \(K((p \lor q) \land \neg Kp \land \neg Kq)\) is subjective

**Lemma: objective, subjective**

Let \( \phi \) be objective. Then \( e, w \models \phi \iff e', w \models \phi \).  
Let \( \sigma \) be subjective. Then \( e, w \models \sigma \iff e, w' \models \sigma \).

When \( \phi \) is objective, “\( w \models \phi \)” stands for “for every \( e, e, w \models \phi \)”.  
When \( \sigma \) is subjective, “\( e \models \sigma \)” stands for “for every \( w, e, w \models \sigma \)”.

Proof on paper
Examples

\( e, w \models K\alpha \iff \text{for all worlds } w', \ w \in e \Rightarrow e, w' \models \alpha \)

Let \( e \overset{\text{def}}{=} \{ w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \} \)
Examples

\[ e, w \models K\alpha \iff \text{for all worlds } w', \ w \in e \Rightarrow e, w' \models \alpha \]

Let \( e \overset{\text{def}}{=} \{ w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \} \)

\( \blacksquare \ w \in e \)
Examples

\[ e, w \models \textbf{K}\alpha \iff \text{for all worlds } w', w \in e \Rightarrow e, w' \models \alpha \]

Let \( e \overset{\text{def}}{=} \{ w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \} \)

\[ \begin{align*}
\blacksquare & w \in e \\
\iff & w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \\
\end{align*} \]
Examples

\[ e, w \models K\alpha \iff \text{for all worlds } w', \ w \in e \Rightarrow e, w' \models \alpha \]

Let \( e \overset{\text{def}}{=} \{ w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \} \)

\[ w \in e \]

\[ \iff w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \]

\[ \iff w \models (p \lor q \lor r) \text{ and } w \models (p \lor q \lor \neg r) \]
Examples

\[ e, w \models K\alpha \iff \text{for all worlds } w', \ w \in e \Rightarrow e, w' \models \alpha \]

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\[ \iff \quad w \models (p \lor q \lor r) \land w \models (p \lor q \lor \neg r) \]
\[ \iff \quad w[p] = 1 \text{ or } w[q] = 1 \text{ or } w[r] = 1, \text{ and} \]
\[ w[p] = 1 \text{ or } w[q] = 1 \text{ or } w[r] = 0 \]
Let $e \overset{\text{def}}{=} \{ w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \}$

- $w \in e$
  - $\iff w \models (p \lor q \lor r) \land (p \lor q \lor \neg r)$
  - $\iff w \models (p \lor q \lor r)$ and $w \models (p \lor q \lor \neg r)$
  - $\iff w[p] = 1$ or $w[q] = 1$ or $w[r] = 1$, and
  - $w[p] = 1$ or $w[q] = 1$ or $w[r] = 0$
  - $\iff w[p] = 1$ or $w[q] = 1$
Examples

\( e, w \models \mathbf{K}\alpha \iff \text{for all worlds } w', \ w \in e \Rightarrow e, w' \models \alpha \)

Let \( e \stackrel{\text{def}}{=} \{ w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \} \)

- \( w \in e \iff w[p] = 1 \text{ or } w[q] = 1 \)
- \( e \models \mathbf{K}(p \lor q) ? \)
Examples

\[ e, w \models \mathbf{K} \alpha \iff \text{for all worlds } w', \ w \in e \Rightarrow e, w' \models \alpha \]

Let \( e \overset{\text{def}}{=} \{ w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \} \)

\begin{itemize}
  \item \( w \in e \iff w[p] = 1 \text{ or } w[q] = 1 \)
  \item \( e \models \mathbf{K}(p \lor q) \ ? \)
    \( \iff \text{for all } w, w \in e \Rightarrow w \models (p \lor q) \)
\end{itemize}
Examples

\[ e, w \models K\alpha \iff \text{for all worlds } w', \ w \in e \Rightarrow e, w' \models \alpha \]

Let \( e \overset{\text{def}}{=} \{ w \mid w \models (p \vee q \vee r) \wedge (p \vee q \vee \neg r) \} \)

1. \( w \in e \iff w[p] = 1 \text{ or } w[q] = 1 \)
2. \( e \models K(p \vee q) \ ? \)
   \( \iff \text{for all } w, w \in e \Rightarrow w \models (p \vee q) \)
   \( \iff \text{for all } w, w \in e \Rightarrow w \models p \text{ or } w \models q \)
Examples

\[ e, w \models K\alpha \iff \text{for all worlds } w', w \in e \Rightarrow e, w' \models \alpha \]

Let \( e \overset{\text{def}}{=} \{w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r)\} \)

\( w \in e \iff w[p] = 1 \text{ or } w[q] = 1 \)

\( e \models K(p \lor q) \quad ? \)

\( \iff \text{for all } w, w \in e \Rightarrow w \models (p \lor q) \)

\( \iff \text{for all } w, w \in e \Rightarrow w \models p \text{ or } w \models q \)

\( \iff \text{for all } w, w \in e \Rightarrow w[p] = 1 \text{ or } w[q] = 1 \)
Examples

\[ e, w \models K\alpha \iff \text{for all worlds } w', \ w \in e \Rightarrow e, w' \models \alpha \]

Let \( e \overset{\text{def}}{=} \{ w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \} \)

- \( w \in e \iff w[p] = 1 \text{ or } w[q] = 1 \)
- \( e \models K(p \lor q) \)
  \[ \iff \text{for all } w, w \in e \Rightarrow w \models (p \lor q) \]
  \[ \iff \text{for all } w, w \in e \Rightarrow w \models p \text{ or } w \models q \]
  \[ \iff \text{for all } w, w \in e \Rightarrow w[p] = 1 \text{ or } w[q] = 1 \]
  \[ \iff \text{for all } w, (w[p] = 1 \text{ or } w[q] = 1) \Rightarrow (w[p] = 1 \text{ or } w[q] = 1) \]
Examples

$e, w \models K\alpha \iff \text{for all worlds } w', w \in e \Rightarrow e, w' \models \alpha$

Let $e \overset{\text{def}}{=} \{ w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \}$

- $w \in e \iff w[p] = 1 \text{ or } w[q] = 1$
- $e \models K(p \lor q) \checkmark$
- $e \models \neg Kp \; ?$
Examples

\[ e, w \models K\alpha \iff \text{for all worlds } w', \ w \in e \Rightarrow e, w' \models \alpha \]

Let \( e \overset{\text{def}}{=} \{ w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \} \)

- \( w \in e \iff w[p] = 1 \text{ or } w[q] = 1 \)
- \( e \models K(p \lor q) \quad \checkmark \)
- \( e \models \neg Kp \quad ? \)
- \( \iff e \not\models Kp \)
Examples

\[ e, w \models K\alpha \iff \text{for all worlds } w', \ w \in e \Rightarrow e, w' \models \alpha \]

Let \( e \overset{\text{def}}{=} \{ w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \} \)

- \( w \in e \iff w[p] = 1 \) or \( w[q] = 1 \)
- \( e \models K(p \lor q) \quad \checkmark \)
- \( e \models \neg Kp \quad ? \)
  \[ \iff e \not\models Kp \]
  \[ \iff \text{for some } w, w \in e \text{ and } w \not\models p \]
Examples

\[ e, w \models K\alpha \iff \text{for all worlds } w', \ w \in e \Rightarrow e, w' \models \alpha \]

Let \( e \overset{\text{def}}{=} \{ w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \} \)

- \( w \in e \iff w[p] = 1 \) or \( w[q] = 1 \)
- \( e \models K(p \lor q) \quad \checkmark \)
- \( e \models \neg Kp \quad ? \)
  \( \iff e \not\models Kp \)
  \( \iff \text{for some } w, w \in e \text{ and } w \not\models p \)
  \( \iff \text{for some } w, w \in e \text{ and } w[p] \neq 1 \)
Examples

\[ e, w \models K\alpha \iff \text{for all worlds } w', w \in e \Rightarrow e, w' \models \alpha \]

Let \( e \overset{\text{def}}{=} \{ w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \} \)

- \( w \in e \iff w[p] = 1 \text{ or } w[q] = 1 \)
- \( e \models K(p \lor q) \quad \checkmark \)
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  \[ \iff \text{for some } w, w[p] = 1 \text{ or } w[q] = 1, \text{ and } w[p] \neq 1 \]
Examples

$e, w \models \text{K} \alpha \iff \text{for all worlds } w', \ w \in e \ \Rightarrow \ e, w' \models \alpha$

Let $e \overset{\text{def}}{=} \{ w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \}$

- $w \in e \iff w[p] = 1 \text{ or } w[q] = 1$
- $e \models \text{K} (p \lor q)$  \[\checkmark\]
- $e \models \neg \text{K} p \ ?$
  - $\iff e \not\models \text{K} p$
  - $\iff \text{for some } w, w \in e \text{ and } w \not\models p$
  - $\iff \text{for some } w, w \in e \text{ and } w[p] \neq 1$
  - $\iff \text{for some } w, w[p] = 1 \text{ or } w[q] = 1, \text{ and } w[p] \neq 1$
  - $\iff \text{for some } w, w[p] \neq 1 \text{ and } w[q] = 1 \ \checkmark$
Examples

\[ e, w \models K\alpha \iff \text{for all worlds } w', \ w \in e \Rightarrow e, w' \models \alpha \]

Let \( e \overset{\text{def}}{=} \{ w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \} \)

- \( w \in e \iff w[p] = 1 \) or \( w[q] = 1 \)
- \( e \models K(p \lor q) \quad \checkmark \)
- \( e \models \neg Kp \quad \checkmark \)
- \( e \models K(p \lor q) \land \neg Kp \land \neg Kq \quad \)?
Examples

\[ e, w \models K\alpha \iff \text{for all worlds } w', \ w \in e \Rightarrow e, w' \models \alpha \]

Let \( e \overset{\text{def}}{=} \{ w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \} \)

\[ \begin{align*}
\ ■ \ w \in e & \iff w[p] = 1 \text{ or } w[q] = 1 \\
\ ■ \ e \models K(p \lor q) & \ \checkmark \\
\ ■ \ e \models \neg Kp & \ \checkmark \\
\ ■ \ e \models K(p \lor q) \land \neg Kp \land \neg Kq & \ ? \\
\ & \iff e \models K(p \lor q) \text{ and } e \models \neg Kp \text{ and } e \models \neg Kq \ \checkmark 
\end{align*} \]
Examples

\[ e, w \models K\alpha \iff \text{for all worlds } w', w \in e \Rightarrow e, w' \models \alpha \]

Let \( e \overset{\text{def}}{=} \{ w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \} \)

- \( w \in e \iff w[p] = 1 \text{ or } w[q] = 1 \)
- \( e \models K(p \lor q) \mkern1.5mu \checkmark \)
- \( e \models \neg Kp \mkern1.5mu \checkmark \)
- \( e \models K(p \lor q) \land \neg Kp \land \neg Kq \mkern1.5mu \checkmark \)
- \( e \models O((p \lor q \lor r) \land (p \lor q \lor \neg r)) \) ?
Examples

\[ e, w \models K\alpha \iff \text{for all worlds } w', \ w \in e \Rightarrow e, w' \models \alpha \]
\[ e, w \models O\alpha \iff \text{for all worlds } w', \ w \in e \Leftrightarrow e, w' \models \alpha \]

Let \( e \overset{\text{def}}{=} \{ w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \} \)

- \( w \in e \iff w[p] = 1 \text{ or } w[q] = 1 \)
- \( e \models K(p \lor q) \checkmark \)
- \( e \models \neg Kp \checkmark \)
- \( e \models K(p \lor q) \land \neg Kp \land \neg Kq \checkmark \)
- \( e \models O((p \lor q \lor r) \land (p \lor q \lor \neg r)) ? \)
  \[ \iff \text{for all } w, w \in e \Leftrightarrow w \models ((p \lor q \lor r) \land (p \lor q \lor \neg r)) \checkmark \]
Examples

$e, w \models K\alpha \iff$ for all worlds $w'$, $w \in e \Rightarrow e, w' \models \alpha$

$e, w \models O\alpha \iff$ for all worlds $w'$, $w \in e \Leftrightarrow e, w' \models \alpha$

Let $e \overset{\text{def}}{=} \{ w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r) \}$

- $w \in e \iff w[p] = 1$ or $w[q] = 1$
- $e \models K(p \lor q)$ ✓
- $e \models \neg Kp$ ✓
- $e \models K(p \lor q) \land \neg Kp \land \neg Kq$ ✓
- $e \models O((p \lor q \lor r) \land (p \lor q \lor \neg r))$ ✓
Logical Omniscience

Logical omniscience means that an agent knows all the consequences of what they know. In particular, they know all valid sentences.
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**Theorem: logical omniscience**

If $\models \alpha \to \beta$, then $\models K\alpha \to K\beta$.

In particular: If $\models \alpha$, then $\models K\alpha$. 
Logical Omniscience

Logical omniscience means that an agent knows all the consequences of what they know. In particular, they know all valid sentences.

**Theorem: logical omniscience**

If $\models \alpha \rightarrow \beta$, then $\models K\alpha \rightarrow K\beta$.

In particular: If $\models \alpha$, then $\models K\alpha$.

Logical omniscience is often problematic:

- Philosophical problem: most agents are not omniscient
- Practical problem: omniscience makes reasoning intractable

We will look at methods to avoid these problems next week.

Proof on paper
The purpose of only-knowing is to capture a knowledge base. Knowledge bases are usually objective. The corresponding epistemic state is then unique:

**Theorem: unique-model property**

Let $\phi$ be objective. Then there is a unique $e$ such that $e \models O\phi$.

An entailment problem $O\phi \models K\alpha$ thus reduces to model checking:

$e \models K\alpha$, where $e = \{w \mid w \models \phi\}$?
Self-Knowledge

We can nest $\mathbf{K}$ operators to say that we know that we know.

Complete and accurate knowledge about own knowledge:

<table>
<thead>
<tr>
<th>Theorem: positive and negative introspection</th>
</tr>
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<tbody>
<tr>
<td>Positive introspection: $\models K\alpha \rightarrow KK\alpha$</td>
</tr>
<tr>
<td>Negative introspection: $\models \neg K\alpha \rightarrow K\neg K\alpha$</td>
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Why?

$e \models (\neg)K\alpha \iff e, w \models (\neg)K\alpha$ for all $w \in e \iff e, w \models K(\neg)K\alpha$. 

Representation Theorem

Can we solve $\mathbf{OKB} \models \alpha$ with ordinary, propositional reasoning?
That is, can we eliminate $\mathbf{K}$ and $\mathbf{O}$?
Then we could use standard reasoning system.
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**Theorem**

Let $\mathbf{KB}$, $\phi$ be objective. Then $\mathbf{OKB} \models \mathbf{K} \phi \iff \mathbf{KB} \models \phi$.

**Idea:** replace nested $\mathbf{K} \phi$ with $\text{TRUE}$ if $\mathbf{KB} \models \phi$, otherwise with $\text{FALSE}$. 
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**Ex.:** Let $\mathbf{KB} \stackrel{\text{def}}{=} (p \lor q \lor r) \land (p \lor q \lor \neg r)$.

$\mathbf{OKB} \models \mathbf{K}((p \lor q) \land \neg \mathbf{K}p \land \neg \mathbf{K}q)$?
Representation Theorem

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$\text{OKB} \models \mathbf{K}((p \lor q) \land \neg \mathbf{K}p \land \neg \mathbf{K}q)$?

$\text{KB} \models p\ \times \quad \text{KB} \models q\ \times$
Representation Theorem

Can we solve $\text{OKB} \models \alpha$ with ordinary, propositional reasoning? That is, can we eliminate $K$ and $O$?
Then we could use standard reasoning system.

Theorem

Let $\text{KB}, \phi$ be objective. Then $\text{OKB} \models K\phi \iff \text{KB} \models \phi$.

Idea: replace nested $K\phi$ with $\text{TRUE}$ if $\text{KB} \models \phi$, otherwise with $\text{FALSE}$.

Ex.: Let $\text{KB} \overset{\text{def}}{=} (p \lor q \lor r) \land (p \lor q \lor \neg r)$.

$\text{OKB} \models K\left( (p \lor q) \land \neg \text{FALSE} \land \neg \text{FALSE} \right)$?
**Representation Theorem**

Can we solve $\text{OKB} \models \alpha$ with ordinary, propositional reasoning? That is, can we eliminate $K$ and $O$? Then we could use standard reasoning system.

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**Idea:** replace nested $K\phi$ with $\text{TRUE}$ if $\text{KB} \models \phi$, otherwise with $\text{FALSE}$.

**Ex.:** Let $\text{KB} \overset{\text{def}}{=} (p \lor q \lor r) \land (p \lor q \lor \neg r)$.

$$\text{KB} \models ((p \lor q) \land \neg \text{FALSE} \land \neg \text{FALSE})? \quad \checkmark$$
Representation Theorem

Can we solve $\text{OKB} \models \alpha$ with ordinary, propositional reasoning? That is, can we eliminate $\text{K}$ and $\text{O}$? Then we could use standard reasoning system.

Theorem

Let $\text{KB}$, $\phi$ be objective. Then $\text{OKB} \models \text{K}\phi \iff \text{KB} \models \phi$.

**Idea:** replace nested $\text{K}\phi$ with $\text{TRUE}$ if $\text{KB} \models \phi$, otherwise with $\text{FALSE}$.

**Ex.:** Let $\text{KB} \overset{\text{def}}{=} (p \lor q \lor r) \land (p \lor q \lor \neg r)$.

$\text{KB} \models ((p \lor q) \land \neg \text{FALSE} \land \neg \text{FALSE})$?  

Next slide formalises this idea.

**Sneak preview:** It’ll become more difficult in the first-order case:
What would you replace $\text{K}Q(x)$ with in $\text{K}\exists x (P(x) \land \neg \text{K}Q(x))$? We’ll see later.
Definition: representation operators

For objective $\mathsf{KB}$ and $\phi$, let $\text{RES}[\mathsf{KB}, \phi] \overset{\text{def}}{=} \begin{cases} \text{TRUE} & \text{if } \mathsf{KB} \models \phi \\ \text{FALSE} & \text{otherwise} \end{cases}$

where \text{TRUE} is some tautology (e.g., $p \lor \neg p$) and \text{FALSE} is $\neg \text{TRUE}$. 
Representation Theorem (2)

**Definition: representation operators**

For objective $\text{KB}$ and $\phi$, let $\text{RES}[\text{KB}, \phi] \overset{\text{def}}{=} \left\{ \begin{array}{ll} \text{TRUE} & \text{if } \text{KB} \models \phi \\ \text{FALSE} & \text{otherwise} \end{array} \right.$

where $\text{TRUE}$ is some tautology (e.g., $p \lor \neg p$) and $\text{FALSE}$ is $\neg \text{TRUE}$.

- $\| P \|^\text{KB} \overset{\text{def}}{=} P$
- $\| \neg \alpha \|^\text{KB} \overset{\text{def}}{=} \neg \| \alpha \|^\text{KB}$
- $\| (\alpha \lor \beta) \|^\text{KB} \overset{\text{def}}{=} (\| \alpha \|^\text{KB} \lor \| \beta \|^\text{KB})$
- $\| K \alpha \|^\text{KB} \overset{\text{def}}{=} \text{RES}[\text{KB}, \| \alpha \|^\text{KB}]$
Representation Theorem (2)

**Definition: representation operators**

For objective KB and φ, let RES[KB, φ] \( \overset{\text{def}}{=} \) \(\begin{cases} \text{TRUE} & \text{if } \text{KB} \models \phi \\ \text{FALSE} & \text{otherwise} \end{cases} \)

where TRUE is some tautology (e.g., \( p \lor \neg p \)) and FALSE is \( \neg \text{TRUE} \).

- \( \|P\|_{KB} \overset{\text{def}}{=} P \)
- \( \|\neg \alpha\|_{KB} \overset{\text{def}}{=} \neg \|\alpha\|_{KB} \)
- \( \|(\alpha \lor \beta)\|_{KB} \overset{\text{def}}{=} (\|\alpha\|_{KB} \lor \|\beta\|_{KB}) \)
- \( \|K \alpha\|_{KB} \overset{\text{def}}{=} \text{RES}[\text{KB}, \|\alpha\|_{KB}] \)

**Theorem: representation theorem**

\( \text{OKB} \models \alpha \iff \models \|\alpha\|_{KB} \).
Example

Let $\text{KB} \overset{\text{def}}{=} (p \lor q \lor r) \land (p \lor q \lor \neg r)$.

$\text{OKB} \models \text{K}( (p \lor q) \land \neg \text{K}p \land \neg \text{K}q)$?

\[
\| K\alpha \|_\text{KB} \overset{\text{def}}{=} \text{RES}[\text{KB}, \| \alpha \|_\text{KB}]
\]

\[
\text{RES}[\text{KB}, \phi] \overset{\text{def}}{=} \text{“KB \models \phi?”}
\]
Example

Let \( \text{KB} \overset{\text{def}}{=} (p \lor q \lor r) \land (p \lor q \lor \neg r) \).

\[ \text{OKB} \models \mathbf{K} \left( (p \lor q) \land \neg \mathbf{K} p \land \neg \mathbf{K} q \right) \]

\[ \iff \models \mathbf{K} \left( (p \lor q) \land \neg \mathbf{K} p \land \neg \mathbf{K} q \right) \models_{\text{KB}} \]

\[
\| \mathbf{K} \alpha \|_{\text{KB}} \overset{\text{def}}{=} \text{RES}[\text{KB}, \| \alpha \|_{\text{KB}}]
\]

\[
\text{RES}[\text{KB}, \phi] \overset{\text{def}}{=} "\text{KB} \models \phi?"
\]
Example

Let $\text{KB} \overset{\text{def}}{=} (p \lor q \lor r) \land (p \lor q \lor \neg r)$.

$\text{OKB} \models K\left((p \lor q) \land \neg Kp \land \neg Kq\right)$

$\iff \models \|K\left((p \lor q) \land \neg Kp \land \neg Kq\right)\|_{\text{KB}}$

$\iff \models \text{RES}[\text{KB}, \|\alpha\|_{\text{KB}}]$
Example

Let $\text{KB} \overset{\text{def}}{=} (p \lor q \lor r) \land (p \lor q \lor \neg r)$.

$\text{OKB} \models K((p \lor q) \land \neg Kp \land \neg Kq)$

$\iff \models \|K((p \lor q) \land \neg Kp \land \neg Kq)\|_{\text{KB}}$

$\iff \models \text{RES}[\text{KB}, \|((p \lor q) \land \neg Kp \land \neg Kq)\|_{\text{KB}}]$.

$\iff \models \text{RES}[\text{KB}, (\langle\langle p \lor q\rangle \land \neg \|Kp\|_{\text{KB}} \land \neg \|Kq\|_{\text{KB}}\rangle)_{\text{KB}}]= p \iff \text{KB} \models p$.

$\iff \models \text{RES}[\text{KB}, (\langle\langle p \lor q\rangle \land \neg \|Kp\|_{\text{KB}} \land \neg \|Kq\|_{\text{KB}}\rangle)_{\text{KB}}]= q \iff \text{KB} \models q$.
Example

Let $KB \overset{\text{def}}{=} (p \lor q \lor r) \land (p \lor q \lor \neg r)$.

$OKB \models K((p \lor q) \land \neg Kp \land \neg Kq)$

\[\iff \models \|K((p \lor q) \land \neg Kp \land \neg Kq)\|_{KB}\]

\[\iff \models \text{RES}[KB, \| (p \lor q) \land \neg Kp \land \neg Kq \|_{KB}]\]

\[\iff \models \text{RES}[KB, ((p \lor q) \land \neg \|Kp\|_{KB} \land \neg \|Kq\|_{KB})]\]

\[\iff \models \text{RES}[KB, ((p \lor q) \land \neg \text{FA,3E} \land \neg \text{FA,3E})]\]

\[\iff \models \text{RES}[KB, (p \lor q) \land \neg \text{FALSE} \land \neg \text{FALSE}]\]

$KB \models (p \lor q) \land \neg \text{FALSE} \land \neg \text{FALSE}$?
Example

Let $KB \overset{\text{def}}{=} (p \lor q \lor r) \land (p \lor q \lor \neg r)$. 

$OKB \models K\left((p \lor q) \land \neg Kp \land \neg Kq\right)$

\[\iff \models \| K\left((p \lor q) \land \neg Kp \land \neg Kq\right) \|_{KB}\]

\[\iff \models \text{RES}[KB, \| (p \lor q) \land \neg Kp \land \neg Kq \|_{KB}]\]

\[\iff \models \text{RES}[KB, ((p \lor q) \land \neg \| Kp \|_{KB} \land \neg \| Kq \|_{KB})]\]

\[\iff \models \text{RES}[KB, ((p \lor q) \land \neg \text{FA,3E} \land \neg \text{FA,3E})]\]

\[\iff \models \text{RES}[KB, ((p \lor q) \land \neg \text{FALSE} \land \neg \text{FALSE})]\]

\[\iff \models \text{TRUE} \quad \checkmark\]
Overview of the Lecture

- A Logic of Knowledge – The Propositional Fragment

- A Logic of Knowledge – The First-Order Case
  - Why first-order logic?
  - Syntax and semantics
  - Knowing that vs knowing what
  - Representation theorem

- Extensions of the Logic of Knowledge
Why Is $\mathcal{OL}_{PL}$ Not Enough?

\[ K( (\spadesuit \lor \heartsuit) \land \neg K \spadesuit \land \neg K \heartsuit ) \]
Why Is $\mathcal{OL}_{PL}$ Not Enough?

$$K \exists x \left( InBox(x) \land \neg KInBox(x) \right)$$
Why Is $\mathcal{OL}_{PL}$ Not Enough?

\[ K \exists x \ (\text{numberOf}(\text{Jane}) = x \land \neg K \text{numberOf}(\text{Jane}) = x) \]
Why Is $\mathcal{OL}_{PL}$ Not Enough?

\[ K \exists x \ (\text{numberOf}(\text{Jane}) = x \land \neg K \text{numberOf}(\text{Jane}) = x) \]

“all” or “some” $\implies$ first-order quantification
The Language of OL

Terms:
- \( x, x', x_1, x_2, \ldots \)  
  first-order variables
- \#1, \#2, \#3, \ldots
  standard names
- \( f(t_1, \ldots, t_j) \)
  functions

Formulas:
- \( P(t_1, \ldots, t_j) \)  
  atomic formulas
- \( t_1 = t_2 \)  
  equality expressions
- \( \exists x \, \alpha \)  
  “for some \( x, \alpha \)”
- \( \forall x \, \alpha \)  
  “for all \( x, \alpha \)”
- \( \neg \alpha \)  
- \( \alpha \lor \beta \)  
- \( \alpha \land \beta \)  
- \( \alpha \rightarrow \beta \)  
- \( \alpha \leftrightarrow \beta \)  
- \( K\alpha \)  
- \( O\alpha \)
The Language of \( OL \)

Terms:

- \( x, x', x_1, x_2, \ldots \) \hspace{1cm} \text{first-order variables}
- \#1, \#2, \#3, \ldots \) \hspace{1cm} \text{standard names}
- \( f(t_1, \ldots, t_j) \) \hspace{1cm} \text{functions}

Formulas:

- \( P(t_1, \ldots, t_j) \) \hspace{1cm} \text{atomic formulas}
- \( t_1 = t_2 \) \hspace{1cm} \text{equality expressions}
- \( \exists x \alpha \) \hspace{1cm} \text{“for some } x, \alpha \text{”}
- \( \forall x \alpha \overset{\text{def}}{=} \neg \exists x \neg \alpha \) \hspace{1cm} \text{“for all } x, \alpha \text{”}
- \( \neg \alpha \) \hspace{1cm} \( \alpha \lor \beta \) \hspace{1cm} \( \alpha \land \beta \) \hspace{1cm} \( \alpha \rightarrow \beta \) \hspace{1cm} \( \alpha \leftrightarrow \beta \) \hspace{1cm} K\alpha \hspace{1cm} O\alpha
Why Standard Names?

Consider in classical logic:
\[
\text{fatherOf}(\text{Sally}) = \text{bestFriend}(\text{Jane}) \land
\text{fatherOf}(\text{Sally}) = \text{bossOf}(\text{John})
\]

- Who is father of Sally?
- “Jane’s best friend” is not a good answer
- “John’s boss” is not a good answer
- Classical logic offers no way of identifying him
- Reason: interpretations \( \langle D, \Phi \rangle \) have different domains

- Standard names correspond to an implicit infinite domain

- Standard names allow to identify individuals in formulas:
  \[
  \text{fatherOf}(\text{Sally}) = \text{Frank}
  \]
The Semantics of $\mathcal{OL}$ (1)

Definition: semantics of $\mathcal{OL}$ (1)

$f(\vec{n})$ or $P(\vec{n})$ are **primitive** iff all $n_i$ are standard names. A term or a formula is **ground** iff it mentions no variable.
The Semantics of $\mathcal{OL}$ (1)

**Definition: semantics of $\mathcal{OL}$ (1)**

- $f(\vec{n})$ or $P(\vec{n})$ are **primitive** iff all $n_i$ are standard names.
- A term or a formula is **ground** iff it mentions no variable.

A *world* $w$ is a function that maps

- primitive functions $f(\vec{n})$ to standard names
- primitive atomic formulas $P(\vec{n})$ to $\{0, 1\}$
The Semantics of $\mathcal{OL}$ (1)

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A **world** $w$ is a function that maps

- primitive functions $f(\vec{n})$ to standard names
- primitive atomic formulas $P(\vec{n})$ to $\{0, 1\}$

The **denotation** of a ground term w.r.t. $w$ is defined as

- $w(n) \overset{\text{def}}{=} n$ for every standard name $n$
- $w(f(t_1, \ldots, t_j)) \overset{\text{def}}{=} w[f(w(t_1), \ldots, w(t_j))]$
The Semantics of $\mathcal{OL}$ (1)

**Definition: semantics of $\mathcal{OL}$ (1)**

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E.g., if Frank is Mia’s father and Mia is Jane’s mother:

\[
\begin{align*}
    w(\text{fatherOf(}w(\text{motherOf(Jane)}))) & = w\left[\text{fatherOf}(w[\text{motherOf(Jane)})])\right] \\
    & = w[\text{fatherOf(Mia)]} \\
    & = \text{Frank.}
\end{align*}
\]
The Semantics of $\mathcal{OL}$ (2)

**Definition: semantics of $\mathcal{OL}$**

An **epistemic state** $e$ is a set of worlds.

- $e, w \models P(t_1, \ldots, t_j) \iff w[P(w(t_1), \ldots, w(t_j))] = 1$
- $e, w \models t_1 = t_2 \iff w(t_1) = w(t_2)$
The Semantics of $\mathcal{OL}$ (2)

Definition: semantics of $\mathcal{OL}$

An epistemic state $e$ is a set of worlds.

- $e, w \models P(t_1, \ldots, t_j) \iff w[P(w(t_1), \ldots, w(t_j))] = 1$
- $e, w \models t_1 = t_2 \iff w(t_1) = w(t_2)$
- $e, w \models \neg \alpha \iff e, w \not\models \alpha$
- $e, w \models (\alpha \lor \beta) \iff e, w \models \alpha$ or $e, w \models \beta$
The Semantics of $\mathcal{OL}$ (2)

**Definition: semantics of $\mathcal{OL}$**

An *epistemic state* $e$ is a set of worlds.

- $e, w \models P(t_1, \ldots, t_j) \iff w[P(w(t_1), \ldots, w(t_j))] = 1$
- $e, w \models t_1 = t_2 \iff w(t_1) = w(t_2)$
- $e, w \models \neg \alpha \iff e, w \not\models \alpha$
- $e, w \models (\alpha \lor \beta) \iff e, w \models \alpha$ or $e, w \models \beta$
- $e, w \models \exists x \alpha \iff e, w \models \alpha^x_n$ for some standard name $n$
The Semantics of $\mathcal{OL}$ (2)

### Definition: semantics of $\mathcal{OL}$

An **epistemic state** $e$ is a set of worlds.

- $e, w \models P(t_1, \ldots, t_j) \iff w[P(w(t_1), \ldots, w(t_j))] = 1$
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- $e, w \models \neg \alpha \iff e, w \not\models \alpha$
- $e, w \models (\alpha \lor \beta) \iff e, w \models \alpha$ or $e, w \models \beta$
- $e, w \models \exists x \alpha \iff e, w \models \alpha^x_n$ for some standard name $n$
- $e, w \models K\alpha \iff$ for all worlds $w'$, $w' \in e \Rightarrow e, w' \models \alpha$
- $e, w \models O\alpha \iff$ for all worlds $w'$, $w' \in e \Leftrightarrow e, w' \models \alpha$
Knowing That vs Knowing What

- $\mathbf{K}\exists x \text{Secret}(x)$  I know *that* some $x$ is a secret
- $\exists x \mathbf{K}\text{Secret}(x)$  I know *which* $x$ is a secret

- $\mathbf{K}\exists x \text{fatherOf}(\text{Sally}) = x$  I know *that* Sally has a father
- $\exists x \mathbf{K}\text{fatherOf}(\text{Sally}) = x$  I know *who* Sally’s father is

- $\mathbf{K}\exists x \alpha = \text{de dicto knowledge}$
- $\exists x \mathbf{K}\alpha = \text{de re knowledge}$

**Theorem: quantifying-in**

\[
\models \forall x \mathbf{K}\alpha \leftrightarrow \mathbf{K}\forall x \alpha \\
\models \exists x \mathbf{K}\alpha \rightarrow \mathbf{K}\exists x \alpha \\
\not\models \mathbf{K}\exists x \alpha \rightarrow \exists x \mathbf{K}\alpha
\]
**Some Properties Inherited From $\mathcal{OL}_PL$**

**Definition: subjective, objective**

If $\phi$ mentions no fun/pred inside $K$ or $O$, we say $\phi$ is **objective**. If $\sigma$ mentions fun/pred only inside $K$ or $O$, we say $\sigma$ is **subjective**.

**Theorem: logical omniscience**

If $\models \alpha \rightarrow \beta$, then $\models K\alpha \rightarrow K\beta$.
If $\models \alpha$, then $\models K\alpha$.

**Theorem: unique-model property**

Let $\phi$ be objective. Then there is a unique $e$ such that $e \models O\phi$.

**Theorem: positive and negative introspection**

Positive introspection: $\models K\alpha \rightarrow KK\alpha$
Negative introspection: $\models \neg K\alpha \rightarrow K\neg K\alpha$
Example

\[
e, w \models \exists x \alpha \iff e, w \models \alpha^x_n \text{ for some standard name } n
\]
\[
e, w \models \mathbf{K} \alpha \iff \text{for all worlds } w', w \in e \Rightarrow e, w' \models \alpha
\]
\[
e, w \models \mathbf{O} \alpha \iff \text{for all worlds } w', w \in e \Leftrightarrow e, w' \models \alpha
\]

Let \( \text{KB} \overset{\text{def}}{=} \exists x (x \neq \#1 \land P(x)) \)
Example

\[ e, w \models \exists x \alpha \iff e, w \models \alpha^n \text{ for some standard name } n \]
\[ e, w \models K\alpha \iff \text{for all worlds } w', w \in e \Rightarrow e, w' \models \alpha \]
\[ e, w \models O\alpha \iff \text{for all worlds } w', w \in e \iff e, w' \models \alpha \]

Let \( KB \overset{\text{def}}{=} \exists x (x \neq #1 \land P(x)) \)

\[ e \models OKB \]
Let $KB \overset{\text{def}}{=} \exists x (x \neq \#1 \land P(x))$

$e \models OKB$

$\iff w \in e \iff w \models \exists x (x \neq \#1 \land P(x))$

$\iff w \in e \iff w[P(n)] = 1$ for some $n \in \{\#2, \#3, \ldots\}$
Example

\[ e, w \models \exists x \alpha \iff e, w \models \alpha^x_n \text{ for some standard name } n \]
\[ e, w \models K\alpha \iff \text{for all worlds } w', w \in e \Rightarrow e, w' \models \alpha \]
\[ e, w \models O\alpha \iff \text{for all worlds } w', w \in e \Leftrightarrow e, w' \models \alpha \]

Let \( KB \overset{\text{def}}{=} \exists x (x \neq \#1 \land P(x)) \)

- \( e \models OKB \)
  \[ \iff w \in e \iff w \models \exists x (x \neq \#1 \land P(x)) \]
  \[ \iff w \in e \iff w[P(n)] = 1 \text{ for some } n \in \{\#2, \#3, \ldots\} \]

- \( e \models K\exists x (P(x) \land \neg KP(x)) \)
Example

\[ e, w \models \exists x \alpha \iff e, w \models \alpha^n_x \text{ for some standard name } n \]
\[ e, w \models K\alpha \iff \text{for all worlds } w', w \in e \Rightarrow e, w' \models \alpha \]
\[ e, w \models O\alpha \iff \text{for all worlds } w', w \in e \iff e, w' \models \alpha \]

Let \( KB \overset{\text{def}}{=} \exists x (x \neq \#1 \land P(x)) \)

- \( e \models OKB \)
  \[ \iff w \in e \iff w \models \exists x (x \neq \#1 \land P(x)) \]
  \[ \iff w \in e \iff w[P(n)] = 1 \text{ for some } n \in \{\#2, \#3, \ldots\} \]

- \( e \models K\exists x (P(x) \land \neg KP(x)) \)
  \[ \iff \text{for all } w, w \in e \Rightarrow \text{for some } n, e, w \models P(n) \land \neg KP(n) \]
Example

\[ e, w \models \exists x \alpha \iff e, w \models \alpha^x_n \text{ for some standard name } n \]
\[ e, w \models K\alpha \iff \text{for all worlds } w', w \in e \Rightarrow e, w' \models \alpha \]
\[ e, w \models O\alpha \iff \text{for all worlds } w', w \in e \Leftrightarrow e, w' \models \alpha \]

Let \( KB \overset{\text{def}}{=} \exists x (x \neq \#1 \land P(x)) \)

- **\( e \models OKB \)**
  \[ \iff w \in e \iff w \models \exists x (x \neq \#1 \land P(x)) \]
  \[ \iff w \in e \iff w[P(n)] = 1 \text{ for some } n \in \{\#2, \#3, \ldots\} \]

- **\( e \models K\exists x (P(x) \land \neg KP(x)) \)**
  \[ \iff \text{for all } w, \ w \in e \Rightarrow \text{for some } n, \ e, w \models P(n) \land \neg KP(n) \]
  \[ \iff \text{for all } w, \ w \in e \Rightarrow \text{for some } n, \ e, w \models P(n) \text{ and } e, w \models \neg KP(n) \]
Example

\[ e, w \models \exists x \alpha \iff e, w \models \alpha^x_n \text{ for some standard name } n \]
\[ e, w \models K\alpha \iff \text{for all worlds } w', w \in e \rightarrow e, w' \models \alpha \]
\[ e, w \models O\alpha \iff \text{for all worlds } w', w \in e \leftrightarrow e, w' \models \alpha \]

Let \( KB \overset{\text{def}}{=} \exists x (x \neq \#1 \land P(x)) \)

- \( e \models OKB \)
  \[ \iff w \in e \Leftrightarrow w \models \exists x (x \neq \#1 \land P(x)) \]
  \[ \iff w \in e \Leftrightarrow w[P(n)] = 1 \text{ for some } n \in \{\#2, \#3, \ldots\} \]

- \( e \models K\exists x (P(x) \land \neg KP(x)) \)
  \[ \iff \text{for all } w, w \in e \Rightarrow \text{for some } n, e, w \models P(n) \land \neg KP(n) \]
  \[ \iff \text{for all } w, w \in e \Rightarrow \text{for some } n, e, w \models P(n) \text{ and } e, w \models \neg KP(n) \]
  \[ \iff \text{for all } w, w \in e \Rightarrow \text{for some } n, e, w \models P(n) \text{ and} \]
  \[ \text{for some } w', w' \in e \text{ and } e, w' \not\models P(n) \]
  \[ \iff \text{for all } w, w \in e \Rightarrow \text{for some } n, w[P(n)] = 1 \text{ and} \]
  \[ \text{for some } w', w' \in e \text{ and } w'[P(n)] \neq 1 \]
Comparison with Tarski Semantics

Traditional FOL semantics

- Interpretation \( \langle D, \Phi \rangle \) plus variable mapping \( \mu \)
- \( \langle D, \Phi \rangle, \mu \models P(t_1, \ldots, t_j) \iff \langle d_1, \ldots, d_j \rangle \in \Phi(P) \) where \( d_i = \langle D, \Phi \rangle, \mu \| t_i \| \)
- \( \langle D, \Phi \rangle, \mu \models \exists x \alpha \iff \langle D, \Phi \rangle, \mu^x_d \models \alpha \) for some \( d \in D \)
- Purpose: reason about mathematics
- Disadvantage: cumbersome to work with

Our semantics

- World maps primitive functions to names, predicates to \( \{0, 1\} \)
- \( w \models P(t_1, \ldots, t_j) \iff w[P(n_1, \ldots, n_j)] = 1 \) where \( n_i = w(t_i) \)
- \( w \models \exists x \alpha \iff w \models \alpha^x_n \) for some standard name \( n \)
- Purpose: reason about knowledge
- Disadvantage: domain is always countably infinite
  - \( \forall x (x = t_1 \lor \ldots \lor x = t_j) \) asserts finite domain in classical FOL
  - \( \forall x (x = t_1 \lor \ldots \lor x = t_j) \) is unsatisfiable in \( \mathcal{OL} \)
  - but can be simulated with predicate: \( \forall x (P(x) \iff (x = t_1 \lor \ldots \lor x = t_j)) \)
  - classical FOL cannot distinguish countably infinite from uncountably infinite domains anyway
Representation Theorem (1)

\[ \text{OKB} \models \exists x \text{KP}(x) \]

How can we represent the known instances of an objective formula?

- \[ \text{KB} \overset{\text{def}}{=} (P(\#1) \land P(\#2)) \quad \text{#1, #2 are known P-instances} \]
- \[ \text{KB} \overset{\text{def}}{=} (P(\#1) \lor P(\#2)) \quad \text{no known P-instances} \]
- \[ \text{KB} \overset{\text{def}}{=} \forall x P(x) \quad \text{all names are known P-instances} \]
- \[ \text{KB} \overset{\text{def}}{=} \forall x (x \neq \#1 \rightarrow P(x)) \quad \text{#2, #3, \ldots are known P-instances} \]
- \[ \text{KB} \overset{\text{def}}{=} (Q(\#1) \land \forall x (Q(x) \rightarrow P(x))) \quad \text{#1 is known P-instance} \]
Representation Theorem (1)

$$\text{OKB } \models \exists x \text{KP}(x)$$

How can we represent the known instances of an objective formula?

- \( \text{KB} \overset{\text{def}}{=} (P(#1) \wedge P(#2)) \) \quad #1, #2 are known \( P \)-instances
- \( \text{KB} \overset{\text{def}}{=} (P(#1) \vee P(#2)) \) \quad no known \( P \)-instances
- \( \text{KB} \overset{\text{def}}{=} \forall x P(x) \) \quad all names are known \( P \)-instances
- \( \text{KB} \overset{\text{def}}{=} \forall x (x \neq #1 \rightarrow P(x)) \) \quad #2, #3, \ldots are known \( P \)-instances
- \( \text{KB} \overset{\text{def}}{=} (Q(#1) \wedge \forall x (Q(x) \rightarrow P(x))) \) \quad #1 is known \( P \)-instance

Let \( n_1, \ldots, n_j \) be names in \( \text{KB} \) and let \( n' \) be a new one.

\[
\text{RES}[\text{KB}, P(x)] \overset{\text{def}}{=} (x = n_1 \wedge "\text{KB } \models P(n_1)") \vee \\
\ldots \\
(x = n_j \wedge "\text{KB } \models P(n_j)") \vee \\
(x \neq n_1 \wedge \ldots \wedge x \neq n_j \wedge "\text{KB } \models P(n')")
\]
Representation Theorem (1)

\[ \text{OKB} \models \exists x \, KP(x)? \]

How can we represent the known instances of an objective formula?

- \( \text{KB} \overset{\text{def}}{=} (P(\#1) \land P(\#2)) \quad x = \#1 \lor x = \#2 \)
- \( \text{KB} \overset{\text{def}}{=} (P(\#1) \lor P(\#2)) \quad \text{FALSE} \)
- \( \text{KB} \overset{\text{def}}{=} \forall x \, P(x) \quad \text{TRUE} \)
- \( \text{KB} \overset{\text{def}}{=} \forall x \, (x \neq \#1 \rightarrow P(x)) \quad x \neq \#1 \)
- \( \text{KB} \overset{\text{def}}{=} (Q(\#1) \land \forall x \, (Q(x) \rightarrow P(x)) \quad x = \#1 \)

Let \( n_1, \ldots, n_j \) be names in \( \text{KB} \) and let \( n' \) be a new one.

\[ \text{RES}[\text{KB}, P(x)] \overset{\text{def}}{=} (x = n_1 \land "\text{KB} \models P(n_1)") \lor \]

\[ \ldots \]

\[ (x = n_j \land "\text{KB} \models P(n_j)") \lor \]

\[ (x \neq n_1 \land \ldots \land x \neq n_j \land "\text{KB} \models P(n')") \]
Representation Theorem (1)

\[ \text{O}_{\text{KB}} \models \exists x \, \text{KP}(x)? \]

How can we represent the known instances of an objective formula?

\[ \text{KB} \overset{\text{def}}{=} (P(#1) \land P(#2)) \quad x = \#1 \lor x = \#2 \]

\[ \text{KB} \overset{\text{def}}{=} (P(#1) \lor P(#2)) \quad \text{FALSE} \]

\[ \text{KB} \overset{\text{def}}{=} \forall x \, P(x) \quad \text{TRUE} \]

\[ \text{KB} \overset{\text{def}}{=} \forall x \, (x \neq \#1 \rightarrow P(x)) \quad x \neq \#1 \]

\[ \text{KB} \overset{\text{def}}{=} (Q(#1) \land \forall x \, (Q(x) \rightarrow P(x)) \quad x = \#1 \]

Let \( n_1, \ldots, n_j \) be names in \( \text{KB} \) and let \( n' \) be a new one.

\[
\text{RES}[\text{KB}, P(x)] \overset{\text{def}}{=} (x = n_1 \land "\text{KB} \models P(n_1)"
\lor \\
\ldots
\lor \\
(x = n_j \land "\text{KB} \models P(n_j)"
\lor \\
(x \neq n_1 \land \ldots \land x \neq n_j \land "\text{KB} \models P(n')")
\]
Representation Theorem (2)

Definition: representation of known instances

If $\phi$ has a free variable $x$ and $n_1, \ldots, n_j$ are the names mentioned in $KB$, $\phi$, and $n'$ is a new name:

$$RES[KB, \phi] \overset{\text{def}}{=} (x = n_1 \land RES[KB, \phi_{n_1}^x]) \lor$$

$$\ldots$$

$$(x = n_j \land RES[KB, \phi_{n_j}^x]) \lor$$

$$(x \neq n_1 \land \ldots \land x \neq n_j \land RES[KB, \phi_{n'}^x]_{n'})$$

If $\phi$ has no free variables:

$$RES[KB, \phi] \overset{\text{def}}{=} \begin{cases} \text{TRUE} & \text{if } KB \models \phi \\ \text{FALSE} & \text{otherwise} \end{cases}$$
Representation Theorem (3)

\[ \| \cdot \| \text{ operator gets a rule for } \exists x \alpha: \]

### Definition: representation operators

- \( \| \phi \|_{KB} \overset{\text{def}}{=} \phi \) for objective \( \phi \)
- \( \| \neg \alpha \|_{KB} \overset{\text{def}}{=} \neg \| \alpha \|_{KB} \)
- \( \| (\alpha \lor \beta) \|_{KB} \overset{\text{def}}{=} (\| \alpha \|_{KB} \lor \| \beta \|_{KB}) \)
- \( \| \exists x \alpha \|_{KB} \overset{\text{def}}{=} \exists x \| \alpha \|_{KB} \)
- \( \| K \alpha \|_{KB} \overset{\text{def}}{=} \text{RES}[KB, \| \alpha \|_{KB}] \)
Representation Theorem (3)

\[ \| \cdot \| \text{ operator gets a rule for } \exists x \alpha: \]

**Definition: representation operators**

- \[ \| \phi \|_{KB} \overset{\text{def}}{=} \phi \text{ for objective } \phi \]
- \[ \| \neg \alpha \|_{KB} \overset{\text{def}}{=} \neg \| \alpha \|_{KB} \]
- \[ \| (\alpha \lor \beta) \|_{KB} \overset{\text{def}}{=} (\| \alpha \|_{KB} \lor \| \beta \|_{KB}) \]
- \[ \| \exists x \alpha \|_{KB} \overset{\text{def}}{=} \exists x \| \alpha \|_{KB} \]
- \[ \| K \alpha \|_{KB} \overset{\text{def}}{=} \text{RES}[KB, \| \alpha \|_{KB}] \]

**Theorem: representation theorem**

\[ \text{OKB} \models \alpha \iff \models \| \alpha \|_{KB}. \]
Overview of the Lecture

- A Logic of Knowledge – The Propositional Fragment
- A Logic of Knowledge – The First-Order Case

- Extensions of the Logic of Knowledge
  - Multiple agents
  - Probabilities
  - Conditional belief
  - Limited Belief (week 8)
  - Actions (week 9)
Mike does not know what is in the gift box, but he knows that Jane knows what is in there:

\[ K_{Mike} \exists x (\text{InBox}(x) \land \neg K_{Mike} \text{InBox}(x) \land K_{Jane} \text{InBox}(x)) \]

Epistemic states get more complex: in every possible world, Mike considers a whole set of worlds to be possible from Jane’s perspective.
I believe that with probability .999, there is no bomb in the gift box:

$$\mathcal{B}(\neg \exists x (\text{InBox}(x) \land \text{Bomb}(x)) : 0.999)$$

An epistemic state is now probability distribution over possible worlds.
Conditional Belief

I believe that if something is in the gift box, it’s probably not a bomb:

$$\mathbf{B} (\exists x \text{InBox}(x) \Rightarrow \neg \text{Bomb}(x))$$

Epistemic state ranks possible worlds by plausibility and checks if the most-plausible worlds where $$\exists x \text{InBox}(x)$$ is true also satisfy $$\text{Bomb}(x)$$.

- A knowledge base is now a collection of conditionals “if ____ , then most likely ____”
- What sort of ranking should these conditionals induce?