Game theory: introduction

1. **Game theory**
   - Introduction to games
   - Representing games
   - Information in games
   - Playing with other rational agents
   - Solving games
   - Zero-sum games
   - Non zero-sum games
Military strategy

Example (Battle of Bismarck)

Battle theatre:
- Two possible routes for Convoy from Rabaul to Lae, each taking three days to complete
- Allies' search aircraft can concentrate on either route
- Bad weather on Northern route makes search difficult
- Once Convoy spotted, bombers deployed to attack it

Decisions:
- Convoy: which route?
- Allies: search where?
Game elements

This setting involves:
- more than one agent (called players): Allies and Convoy
- moves/strategies for each player: choice of route for convoy; search area for Allies
- outcomes that co-depend on the strategies of both players (play): the four possible scenarios above
- preferences over outcomes represented by payoffs for each player: number of days convoy is bombed

Game theory

Definition (Game)

A game is any setting in which there are more than one decision-makers, called players, and in which the outcomes may co-depend on the actions/strategies of all players.

- A solution of the game is any combination of rational strategies by all players

Aim of Game Theory

The aim of game theory is to identify solutions to games.
Strategic analysis

Allies' option 1: Concentrate search in the North.

Kenney Strategy: Concentrate reconnaissance on northern route.
Estimated Outcome: Although reconnaissance would be hampered by poor visibility, the convoy should be discovered by the second day, which would permit two days of bombing.

Two Days of Bombing

Allies' option 2: Concentrate search to the South.

Kenney Strategy: Concentrate reconnaissance on southern route.
Estimated Outcome: With poor visibility and limited reconnaissance, Kenney could not expect the convoy to be discovered until it broke out into clear weather on the third day. This would permit only one day of bombing.

One Day of Bombing

Kenney Strategy: Concentrate reconnaissance on southern route.
Estimated Outcome: With good visibility and concentrated reconnaissance in the area, the convoy should be sighted almost as soon as it sailed from Rabaul. This would allow three days of bombing.

Three Days of Bombing
**Battle: table representation**

<table>
<thead>
<tr>
<th>N</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>Allies</td>
<td>Convoy</td>
</tr>
<tr>
<td>N</td>
<td>2</td>
</tr>
<tr>
<td>S</td>
<td>1</td>
</tr>
</tbody>
</table>

- **Rows** = Allies’ general’s (A) actions/strategies
- **Columns** = Convoy’s general’s (C) actions
- Abstraction: row player and column player
- Payoffs: number of bombing days

**Game theory FAQ**

**Question**

Haven’t we seen this already?

**What’s new/different?**

- One source of uncertainty due to others’ strategies . . .
- Information about others’ preferences
- Additional information: assumption that other agents rational!
Example (Pursue and evade)

A prisoner (P) is planning an escape from prison. There are two possible escape routes: in the prison’s North or South wings. A prison guard (G) is on watch. The guard can patrol one wing but not both.

Prison escape: game tree analysis

- Represent this scenario as a game tree:

  From G’s viewpoint:

  ![Game tree diagram]

  where in outcome e the prisoner escapes, and in c he’s caught.

- Each player has different payoff, or utility, functions for the outcomes; for each player $p$ (here $p \in \{P, G\}$):

  $$u_p : \Omega \to \mathbb{R}$$
Prison escape: game tree

- Combine each player’s payoffs:

- Each outcome has a *payoff vector*, one value for each player:

$$ (u_P(\omega), u_G(\omega)) $$

- In this case payoffs are complementary: *i.e.*, $u_P(\omega) + u_G(\omega) = 0$. Such games are called *zero-sum games*.

Games in extensive form

**Definition (Game tree)**

A *game tree* is also called the *extensive form* of a game.

Game trees allow fine modelling of games:

- *individual moves* at different stages for each player
- *turn structure*: players make moves at different stages: e.g., alternating, simultaneous, etc.
- *information states*, or *epistemic states* (*states of knowledge*), of players at each decision point
- *contingent/conditional actions/strategies* for each player which depend on its epistemic state: e.g., if prisoner moves North, I’ll move North too.
Prison escape: epistemic state

Case 1: Guard observes prisoner’s movements:

- Additional knowledge/information about the prisoner’s move gives guard an advantage
- Guard’s optimal strategy: “follow prisoner’s move”; i.e., if P moves n, then move N; if P moves s, then move S

Escape reversed

Case 2: Prisoner observes guard’s movements:

- Additional knowledge gives advantage to the prisoner
- Prisoner’s optimal strategy: move opposite the guard; i.e., if G moves N, then move s; if G moves S, then move n.
Modelling information

Case 3: Neither observes the other’s move (e.g., simultaneous moves):

![Game Graph]

Definition (Information set)

An information set is a set of decision nodes that are epistemically indistinguishable by an agent. An information set defines an agent’s epistemic state at some decision point. In a game of perfect information every information set has only a single node; i.e., is a singleton set.

Epistemic modelling

- The game graph on the right is an alternative representation of prisoner escape game in Case 3
- Here P’s action is unknown to G: i.e., both possibilities lead to same epistemic state for G
- G’s moves are non-deterministic in sense that same action leads to different outcomes
Normal form

Definition

A game matrix is called the *normal (strategic) form* of a game.

What do the normal forms of the game trees above look like?

<table>
<thead>
<tr>
<th>G</th>
<th>N</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>P n</td>
<td>−1,1</td>
<td>1,−1</td>
</tr>
<tr>
<td>s</td>
<td>1,−1</td>
<td>−1,1</td>
</tr>
</tbody>
</table>

Modelling information

- By observing P’s move in Case 1, G should have a ‘winning strategy’; *i.e.*, one that always yields payoff 1 to G
- Let F be guard’s optimal strategy: “follow prisoner’s move”
Possible strategies

Definition

A strategy for an agent is the specification of a unique move in each of its (reachable) information sets (epistemic states).

Possible strategies for G in Case 1:

- if n, then N; if s, then N
- if n, then N; if s, then S
- if n, then S; if s, then N
- if n, then S; if s, then S

Normal form

Victor Jauregui | Engineering Decisions
Meet Alice and Bob

Example: Alice and Bob

Example (Alice, Bob, and a coconut)

Alice (A) and Bob (B) are at a coconut tree which has only one coconut worth 10 kilocalories (kc) of energy in total. To get the coconut, one (or both) must climb the tree to shake it loose. If must be dropped from the top of the tree to crack it open. It would take Alice some effort (2kc) to climb the tree, whereas Bob’s effort is negligible.

If Bob climbs (c) the tree and Alice waits (W) below then Alice will get to the coconut first, eating most of it (9kc worth) and leaving only a small portion for Bob. If Alice climbs (C) and Bob waits (w) below then Bob will get to it first and eat his fill (4kc worth) before Alice gets down and takes it off him. If both climb up, Bob will climb down quicker and eat some (3kc worth) before Alice gets down and takes the rest.
Game structure: Alice moves first

- Suppose Alice moves first; in which case Bob will gain information about Alice’s move.

```
\begin{array}{c|cc}
 & c & w \\
\hline
C & 5 & 3 \\
B & 4 & 4 \\
A & 9 & 1 \\
W & 0 & 0 \\
\end{array}
```

- What should Alice do?
  - Wait below hoping for 9kc and risk 0kc?
  - Climb herself, settling for something in between?

Games vs single-agent decisions

- From Alice's perspective the ‘decision table’ would look like the one above.
- Alice might use one of the decision rules under ignorance as she doesn’t know what Bob will do; e.g., Maximin (C).
- But Alice isn’t ignorant about Bob! Alice knows Bob is rational (i.e., will try to maximise utility).
Alice’s ‘What if ...’ analysis

Alice’s conclusion

Alice’s best strategy, considering Bob’s rational response, should be to Wait in preference to Climbing (payoff to Alice of 9 compared to 4).

Victor Jauregui
Engineering Decisions

Strategies and counter-strategies

- If Alice moves first, Bob has more information, and hence more strategic options; i.e., Bob’s possible pure strategies are:
  - Regardless of whether Alice climbs or waits, I will wait
  - Regardless of whether Alice climbs or waits, I will climb
  - I will do the same as Alice: i.e., if Alice climbs, I will climb; if Alice waits I will wait
  - I will do the opposite of Alice: i.e., if Alice climbs, I will wait; if Alice waits I will climb
- If Alice waits, then Bob’s best counter-strategy is to climb
- If Alice climbs, then Bob’s best counter-strategy is to wait
- Combining these, Bob’s optimal strategy is to do the opposite of what Alice does
### Additional information of games

- **Game matrix:**

<table>
<thead>
<tr>
<th></th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>C/w</td>
<td>W/w</td>
</tr>
<tr>
<td>C/c</td>
<td>C/c</td>
</tr>
<tr>
<td>W</td>
<td>W/c</td>
</tr>
<tr>
<td>W</td>
<td>W/c</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>A</th>
<th>C</th>
<th>W/w</th>
<th>W</th>
<th>W/c</th>
<th>W/c</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>4,4</td>
<td>5,3</td>
<td>4,4</td>
<td>5,3</td>
<td></td>
</tr>
<tr>
<td>W</td>
<td>0,0</td>
<td>0,0</td>
<td>9,1</td>
<td>9,1</td>
<td></td>
</tr>
</tbody>
</table>

- Bob’s dominant strategy is: “do the opposite of what Alice does”; *i.e.*, “if Alice waits, then I climb; if Alice climbs, I wait”: W/c

### Reasoning about other agents’ preferences

- Previous example shows why multi-agent decisions are more complex than single agent decisions
- Epistemic states affect available strategies
- Multi-agent decisions should incorporate the preferences and epistemic state of the other agents; *e.g.*, Alice’s “what if . . .” analysis of Bob’s response to her move

### Conclusion

Reasoning about other players’ preferences might improve the outcome for either or both players.
Game solutions

Definition (Plays and solutions)

In two-player games, a play is a pair \((s_1, s_2)\) consisting of a strategy for each player. A play uniquely determines an outcome to the game. For \(n\)-player games this generalises to \(n\)-tuples \((s_1, s_2, \ldots, s_n)\). The outcome of ‘rational’ strategies from each player is called a solution to the game.

- Game theory is about developing methods and techniques to identify solutions to games
- Dominance can help simplify the problem based on the agents’ preferences
- Do all games have solutions? (Existence)
- Can a game have more than one solution? (Uniqueness)

Two-player strictly competitive games

Definition (Two-player strictly competitive game)

A two-player strictly competitive (adversarial) game is one in which the preferences of each agent are in opposition. A zero-sum game is a strictly competitive game in which the agents’ payoffs are complementary; i.e., their sum is zero.

- For example:

<table>
<thead>
<tr>
<th>(s)</th>
<th>(r)</th>
<th>(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>(-1,1)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>(S)</td>
<td>(0,0)</td>
<td>(2,-2)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(s)</th>
<th>(r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(S)</td>
<td>(0)</td>
</tr>
</tbody>
</table>

  Other examples: chess, poker, football, etc.
- Because payoffs complementary, by convention only row player’s shown
Dominance-based solutions

Recall that:

**Definition (Dominance)**

A strategy $A$ is *dominated* by strategy $B$ if for each of the other player’s strategies, the outcome of $B$ is at least as preferred as that of the corresponding outcome of $A$, and for some strategy of the other player it is strictly more preferred.

$$
\begin{array}{ccc}
  & a & b & c \\
A & 1 & 2 & 4 \\
B & 3 & 2 & 5 \\
\end{array}
$$

- If $A$ is dominated by $B$, then $B$ is a *better strategy* regardless of what strategy player 2 plays; *i.e.*, it is a universally *better response*.
- Dominated strategies can be disregarded/discarded.

Dominance helps find solutions by eliminating strategies that neither player will play.

**Exercise**

Apply dominance to simplify the following game by eliminating dominated strategies.

$$
\begin{array}{ccc}
  & a & b & c \\
A & 1 & 2 & 4 \\
B & 3 & 2 & 5 \\
\end{array}
$$

- Dominance helps find solutions by eliminating strategies that neither player will play.
- The plays left after dominance in the game above are $(B,a)$ and $(B,b)$—are these satisfactory solutions?
The battle of the Bismarck Sea

- The battle of the Bismarck Sea is a zero-sum game with imperfect information (neither the convoy Captain nor Allies’ General know the other’s move).

- Payoffs are assumed to be complementary

\[
\begin{array}{c|cc}
\text{Convoy} & n & s \\
\hline
\text{Allies} & N & 2 & 2 \\
& S & 1 & 3 \\
\end{array}
\]

- Accordingly, the column player prefers outcomes with smaller values in the table.

- The Battle of the Bismarck Sea is *iterated dominance solvable*. 
Meet Alice and Bob

The coconut game is not zero-sum:

<table>
<thead>
<tr>
<th></th>
<th>W/c</th>
<th>W/w</th>
<th>W/c</th>
<th>W/c</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>W</td>
<td>9, 1</td>
<td>0, 0</td>
<td>9, 1</td>
<td>9, 1</td>
</tr>
<tr>
<td>C</td>
<td>5, 3</td>
<td>4, 4</td>
<td>5, 3</td>
<td>5, 3</td>
</tr>
</tbody>
</table>

Dominance implies that Alice should Wait, Bob should do the opposite of Alice (i.e., climb if Alice waits, and wait if Alice climbs); compare with Maximin (C) which has a value of 4
Reversing roles

- What if Bob moves first?

```
  B  w  0,0  0,0  4,4  4,4
   c  1,9  3,5  1,9  3,5
  A  w  0,0  0,0  4,4  4,4
   c  1,9  3,5  1,9  3,5
  C  3,5
  W  0,0
  A  1,9
  W  0,0
```

- By moving first Bob causes Alice to climb!

Summary

- Behaviour of other rational agents makes multi-agent decisions more complex:
  - information about other agents’ preferences
  - assumption of rationality
- Games can be represented in normal (table/matrix) and extensive (tree) forms
- Zero-sum (constant sum) games: e.g., Bismarck Sea battle
- Non-zero-sum games: e.g., Coconut game
- Used multi-lateral (iterated) dominance to narrow-in on admissible solutions