Exercise sheet 2a – Solutions and Hints
COMP6741: Parameterized and Exact Computation
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Exercise 1. A dominating set of a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that $N_G[S] = V$.

**Degree-5 Dominating Set**
- **Input:** A graph $G = (V, E)$ with maximum degree at most 5 and an integer $k$
- **Parameter:** $k$
- **Question:** Does $G$ have a dominating set of size at most $k$?

Design a linear kernel for Degree-5 Dominating Set.
**Solution sketch.** Simplification rule: If $|V| > 6 \cdot k$, then return No.

Exercise 2. Consider the following problem.

**Point Line Cover**
- **Input:** A set of points $P$ in $\mathbb{Z}^2$, and an integer $k$
- **Parameter:** $k$
- **Question:** Is there a set $L$ of at most $k$ lines in $\mathbb{R}^2$ such that each point in $P$ lies on at least one line in $L$?

Example: $(P = \{(-1, -2), (0, 0), (1, -1), (1, 1), (1, 2), (1, 3), (1, 4), (2, 4)\}, k = 2)$ is a Yes-instance since the lines $x = 1$ and $y = 2x$ cover all the points.

Show that Point Line Cover has a polynomial kernel.
**Solution sketch.**
(1) First, we show that the algorithm can restrict its attention to $O(|P|^2)$ candidate lines. Indeed, if a set of lines cover all the points and this set contains a line covering only one point, we can replace it by a line covering at least 2 points.
(2) Second, we design a simplification rule for the case where one candidate line covers many points in $P$: If some candidate line covers more than $k$ points, then this line is part of the solution, because any other line covers at most one of these points.
(3) Finally, we design a simplification rule that solves Point Line Cover when $|P|$ is large compared to $k$: If there are more than $k^2$ points left after having exhaustively applied the previous rule, we have a No-instance since all remaining candidate lines cover at most $k$ points.

Exercise 3. A cluster graph is a graph where every connected component is a complete graph.
Cluster Editing

Input: Graph $G = (V, E)$, integer $k$
Parameter: $k$

Question: Is it possible to edit (add or delete) at most $k$ edges of $G$ so that it becomes a cluster graph?

1. Show that $G$ is a cluster graph iff $G$ contains no induced $P_3$ (path with 3 vertices).

2. Design a kernel for Cluster Editing with $O(k^2)$ vertices.

Hint. Design simplification rules for (1) a vertex that does not occur in any $P_3$, (2) an edge that occurs in many $P_3$s, and (3) a non-edge that occurs in many $P_3$s

Solution sketch.

1. $G$ is a cluster graph iff it has no induced $P_3$
   - $(\Rightarrow)$ Consider any three vertices of a cluster graph and make a case analysis of how they are connected.
   - $(\Leftarrow)$ Build a new cluster by starting from an arbitrary vertex. As long as there is a vertex that is connected to a vertex in the current cluster, first, observe that it is connected to all the vertices in the current cluster, and then add it to the current cluster. When no vertex has a neighbor in the current cluster, we have finished the current cluster; it is a clique; and we can start the next cluster of the graph, until all vertices have been assigned to a cluster.

2. Simplification rules. Note that we can list all induced $P_3$’s in polynomial time.
   - If there is a vertex that belongs to no induced $P_3$, then remove that vertex.
   - If some edge belongs to more than $k$ induced $P_3$’s, then delete that edge and set $k \leftarrow k - 1$.
   - If some non-edge $uv \notin E$ belongs to more than $k$ induced $P_3$’s, then add the edge $uv$ to $G$ and set $k \leftarrow k - 1$.
   - If there are more than $k^2$ induced $P_3$’s left and none of the previous rules applies, then return No.

If no simplification rule applies, we have a graph on at most $3k^2$ vertices.