7. Parameter Treewidth
COMP6741: Parameterized and Exact Computation

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19T3

Outline

1. Algorithms for trees
2. Tree decompositions
3. Monadic Second Order Logic
4. Dynamic Programming over Tree Decompositions
   - SAT
   - CSP
5. Further Reading

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Exercise

Recall: An independent set of a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that $G[S]$ has no edge.

#INDEPENDENT SETS ON TREES	
Input:	A tree $T = (V, E)$
Output:	The number of independent sets of T.

- Design a polynomial time algorithm for #INDEPENDENT SETS ON TREES

Exercise

Recall: A dominating set of a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that $N_G[S] = V$.

#Dominating Sets on Trees	
Input:	A tree $T = (V, E)$
Output:	The number of dominating sets of T.

- Design a polynomial time algorithm for #Dominating Sets on Trees

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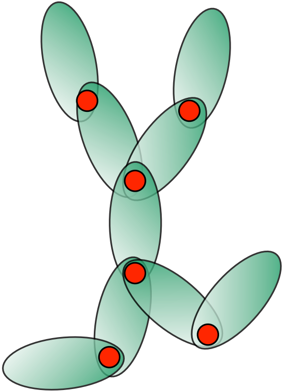
Algorithms using graph decompositions

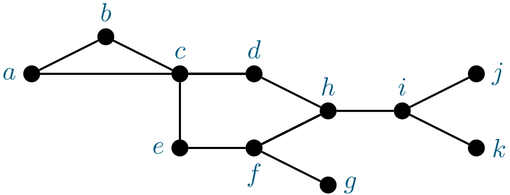


Idea: decompose the problem into subproblems and combine solutions to subproblems to a global solution.

Parameter: overlap between subproblems.

Tree decompositions (by example)

- A graph G



Tree decompositions (by example)

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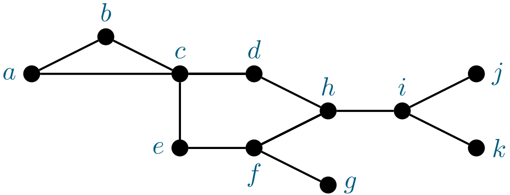
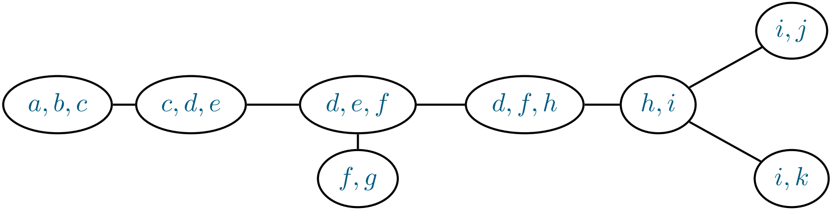


- A *tree decomposition* of G



Tree decompositions (by example)

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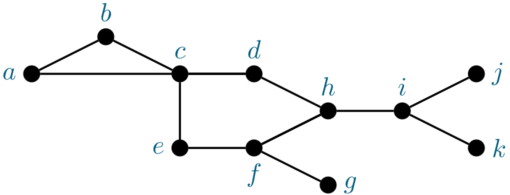

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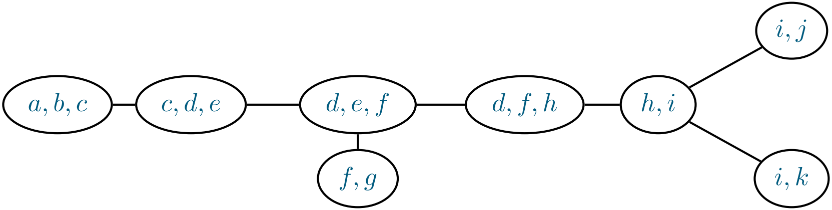
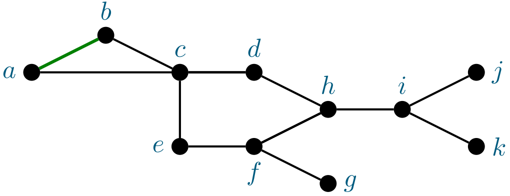


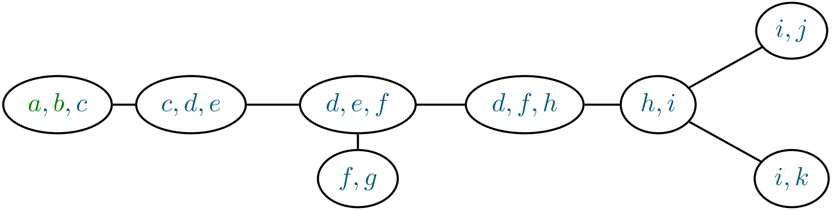
Conditions:

Tree decompositions (by example)

- A graph G



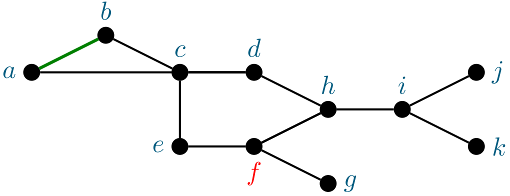
- A *tree decomposition* of G



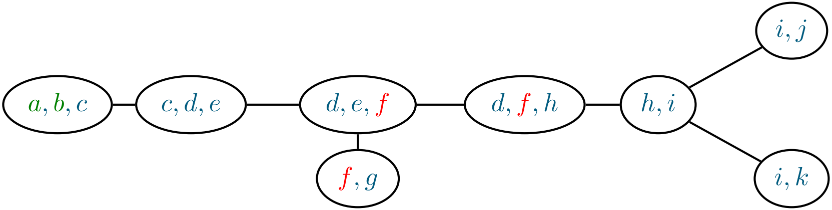
Conditions: covering

Tree decompositions (by example)

- A graph G



- A *tree decomposition* of G



Conditions: covering and connectedness.

Tree decomposition (more formally)

- Let G be a graph, T a tree, and γ a labeling of the vertices of T by sets of vertices of G.
- We refer to the vertices of T as "nodes", and we call the sets $\gamma(t)$ "bags".
- The pair (T, γ) is a *tree decomposition* of G if the following three conditions hold:
  1. For every vertex v of G there exists a node t of T such that $v \in \gamma(t)$.
  2. For every edge vw of G there exists a node t of T such that $v, w \in \gamma(t)$ ("covering").
  3. For any three nodes t_1, t_2, t_3 of T, if t_2 lies on the unique path from t_1 to t_3, then $\gamma(t_1) \cap \gamma(t_3) \subseteq \gamma(t_2)$ ("connectedness").

Treewidth

- The *width* of a tree decomposition (T, γ) is defined as the maximum $|\gamma(t)| - 1$ taken over all nodes t of T.
- The *treewidth* $\text{tw}(G)$ of a graph G is the minimum width taken over all its tree decompositions.

Basic Facts

- Trees have treewidth 1.
- Cycles have treewidth 2.
- Consider a tree decomposition (T, γ) of a graph G and two adjacent nodes i, j in T. Let T_i and T_j denote the two trees obtained from T by deleting the edge ij, such that T_i contains i and T_j contains j. Then, every vertex contained in both $\bigcup_{a \in V(T_i)} \gamma(a)$ and $\bigcup_{b \in V(T_j)} \gamma(b)$ is also contained in $\gamma(i) \cap \gamma(j)$.
- The complete graph on n vertices has treewidth $n - 1$.
- If a graph G contains a clique K_r, then every tree decomposition of G contains a node t such that $K_r \subseteq \gamma(t)$.

Complexity of Treewidth

TREEWIDTH
Input: Graph $G = (V, E)$, integer k
Parameter: k
Question: Does G have treewidth at most k?

- TREEWIDTH is NP-complete.
- TREEWIDTH is FPT: there is a $k^{O(k^3)} \cdot |V|$ time algorithm [Bod96]

Easy problems for bounded treewidth

- Many graph problems that are polynomial time solvable on trees are FPT with parameter treewdith.
- Two general methods:
  - *Dynamic programming*: compute local information in a bottom-up fashion along a tree decomposition
  - *Monadic Second Order Logic*: express graph problem in some logic formalism and use a meta-algorithm

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Monadic Second Order Logic

- **Monadic Second Order (MSO)** Logic is a powerful formalism for expressing graph properties. One can quantify over vertices, edges, vertex sets, and edge sets.
- **Courcelle's theorem** [**Cour.celle90**]. Checking whether a graph G satisfies an MSO property is FPT parameterized by the treewidth of G plus the length of the MSO expression.
- **Arnborg et al.'s generalizations** [ALS91].
  - FPT algorithm for parameter $\mathsf{tw}(G) + |\phi(X)|$ that takes as input a graph G and an MSO sentence $\phi(X)$ where X is a free (non-quantified) vertex set variable, that computes a minimum-sized set of vertices X such that $\phi(X)$ is true in G.
  - Also, the input vertices and edges may be colored and their color can be tested.

Elements of MSO

An MSO formula has

- variables representing vertices $(u, v, \ldots)$, edges $(a, b, \ldots)$, vertex subsets $(X, Y, \ldots)$, or edge subsets $(A, B, \ldots)$ in the graph
- atomic operations
  - $u \in X$: testing set membership
  - $X = Y$: testing equality of objects
  - $\mathsf{inc}(u, a)$: incidence test "is vertex u an endpoint of the edge a?"
- propositional logic on subformulas: $\phi_1 \wedge \phi_2$, $\phi_1 \vee \phi_2$, $\neg\phi_1$, $\phi_1 \Rightarrow \phi_2$
- Quantifiers: $\forall X \subseteq V$, $\exists A \subseteq E$, $\forall u \in V$, $\exists a \in E$, etc.

Shortcuts in MSO

We can define some shortcuts

- $u \neq v$ is $\neg(u = v)$
- $X \subseteq Y$ is $\forall v \in V.\ (v \in X) \Rightarrow (v \in Y)$
- $\forall v \in X\ \varphi$ is $\forall v \in V.\ (v \in X) \Rightarrow \varphi$
- $\exists v \in X\ \varphi$ is $\exists v \in V.\ (v \in X) \wedge \varphi$
- $\mathsf{adj}(u, v)$ is $(u \neq v) \wedge \exists a \in E.\ (\mathsf{inc}(u, a) \wedge \mathsf{inc}(v, a))$

MSO Logic Example

Example: 3-$\textsc{Coloring}$,

- "there are three independent sets in $G = (V, E)$ which form a partition of V"
- $$\begin{aligned}\mathsf{3COL} := \exists R \subseteq V.\ \exists G \subseteq V.\ \exists B \subseteq V.\ & \\ & \mathsf{partition}(R, G, B) \\ & \wedge \mathsf{independent}(R) \wedge \mathsf{independent}(G) \wedge \mathsf{independent}(B),\end{aligned}$$

  where

  $$\begin{aligned}\mathsf{partition}(R, G, B) := \forall v \in V.\ ((v \in R \wedge v \notin G \wedge v \notin B) & \\ \vee (v \notin R \wedge v \in G \wedge v \notin B) \vee (v \notin R \wedge v \notin G \wedge v \in B))\end{aligned}$$

  and

  $$\mathsf{independent}(X) := \neg(\exists u \in X.\ \exists v \in X.\ \mathsf{adj}(u, v))$$

MSO Logic Example II

By Courcelle's theorem and our 3COL MSO formula, we have:

Theorem 1

3-COLORING *is* FPT *with parameter treewidth.*

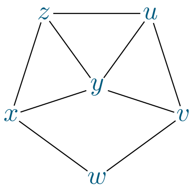
Treewidth only for graph problems?

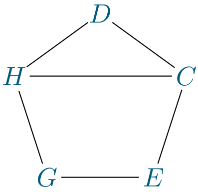
Let us use treewidth to solve a Logic Problem

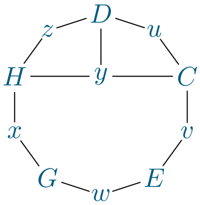
- associate a graph with the instance
- take the tree decomposition of the graph
- most widely used: primal graphs, incidence graphs, and dual graphs of formulas.

Three Treewidth Parameters

CNF Formula $F = C \wedge D \wedge E \wedge G \wedge H$ where $C = (u \vee v \vee \neg y)$, $D = (\neg u \vee z \vee y)$, $E = (\neg v \vee w)$, $G = (\neg w \vee x)$, $H = (x \vee y \vee \neg z)$.







primal graph dual graph incidence graph

This gives rise to parameters primal treewidth, dual treewidth, and incidence treewidth.

Formally

Definition 2

Let F be a CNF formula with variables $\mathsf{var}(F)$ and clauses $\mathsf{cla}(F)$.
The primal graph of F is the graph with vertex set $\mathsf{var}(F)$ where two variables are adjacent if they appear together in a clause of F.
The dual graph of F is the graph with vertex set $\mathsf{cla}(F)$ where two clauses are adjacent if they have a variable in common.
The incidence graph of F is the bipartite graph with vertex set $\mathsf{var}(F) \cup \mathsf{cla}(F)$ where a variable and a clause are adjacent if the variable appears in the clause.
The primal treewidth, dual treewidth, and incidence treewidth of F is the treewidth of the primal graph, the dual graph, and the incidence graph of F, respectively.

Incidence treewidth is most general

Lemma 3

The incidence treewidth of F is at most the primal treewidth of F plus 1.

Proof.

Start from a tree decomposition (T, γ) of the primal graph with minimum width. For each clause C:

- There is a node t of T with $\mathsf{var}(C) \subseteq \gamma(t)$, since $\mathsf{var}(C)$ is a clique in the primal graph.
- Add to t a new neighbor t' with $\gamma(t') = \gamma(t) \cup \{C\}$.

$\square$

Incidence treewidth is most general II

Lemma 4

The incidence treewidth of F is at most the dual treewidth of F plus 1.

Incidence treewidth is most general II

Lemma 4

The incidence treewidth of F is at most the dual treewidth of F plus 1.

Primal and dual treewidth are incomparable.

- One big clause alone gives large primal treewidth.
- $\{\{x, y_1\}, \{x, y_2\}, \ldots, \{x, y_n\}\}$ gives large dual treewidth.

SAT parameterized by treewidth

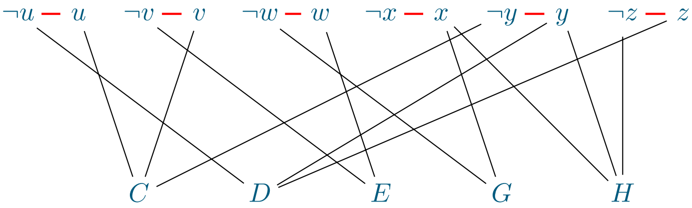
SAT
Input: A CNF formula F
Question: Is there an assignment of truth values to var(F) such that F evaluates to true?

Note: If SAT is FPT parameterized by incidence treewidth, then SAT is FPT parameterized by primal treewidth and by dual treewidth.

SAT is FPT for parameter incidence treewidth

CNF Formula $F = C \wedge D \wedge E \wedge G \wedge H$ where $C = (u \vee v \vee \neg y)$, $D = (\neg u \vee z \vee y)$, $E = (\neg v \vee w)$, $G = (\neg w \vee x)$, $H = (x \vee y \vee \neg z)$

Auxiliary graph:



- MSO Formula: *"There exists an independent set of literal vertices that dominates all the clause vertices."*
- The treewidth of the auxilary graph is at most twice the treewidth of the incidence graph plus one.

FPT via MSO

Theorem 5

SAT *is* FPT *for each of the following parameters: primal treewidth, dual treewidth, and incidence treewidth.*

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Coucelle's theorem: discussion

Advantages of Courcelle's theorem:

- general, applies to many problems
- easy to obtain FPT results

Drawback of Courcelle's theorem

- the resulting running time depends non-elementarily on the treewidth t and the length ℓ of the MSO-sentence, i.e., a tower of 2's whose height is $\omega(1)$

$$2^{2^{2^{\cdot^{\cdot^{\cdot^{t+\ell}}}}}}$$

Dynamic progamming over tree decompositions

Idea: extend the algorithmic methods that work for trees to tree decompositions.

- Step 1 Compute a minumum width tree decomposition using Bodlaender's algorithm
- Step 2 Transform it into a standard form making computations easier
- Step 3 Bottom-up Dynamic Programming (from the leaves of the tree decomposition to the root)

Nice tree decomposition

A *nice* tree decomposition (T, γ) has 4 kinds of bags:

- *leaf node*: leaf t in T and $|\gamma(t)| = 1$
- *introduce node*: node t with one child t' in T and $\gamma(t) = \gamma(t') \cup \{x\}$
- *forget node*: node t with one child t' in T and $\gamma(t) = \gamma(t') \setminus \{x\}$
- *join node*: node t with two children t_1, t_2 in T and $\gamma(t) = \gamma(t_1) = \gamma(t_2)$

Every tree decomposition of width w of a graph G on n vertices can be transformed into a nice tree decomposition of width w and $O(w \cdot n)$ nodes in polynomial time [Klo94].

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Dynamic programming: primal treewidth

- Compute a nice tree decomposition (T, γ) of F's primal graph with minimum width [Bod96; Klo94]
- Select an arbitary root r of T
- Denote T_t the subtree of T rooted at t
- Denote $\gamma_\downarrow(t) = \{x \in \gamma(t') : t' \in V(T_t)\}$
- Denote $F_\downarrow(t) = \{C \in F : \mathsf{var}(C) \subseteq \gamma_\downarrow(t)\}$
- For a node t and an assignment $\tau : \gamma(t) \to \{0,1\}$, define

$$\mathsf{sat}(t,\tau) = \begin{cases} 1 & \text{if } \tau \text{ can be extended to a} \\ & \text{satisfying assignment of } F_\downarrow(t) \\ 0 & \text{otherwise.} \end{cases}$$

DP: primal treewidth II

$$\mathsf{sat}(t,\tau) = \begin{cases} 1 & \text{if } \tau \text{ can be extended to a} \\ & \text{satisfying assignment of } F_{\downarrow}(t) \\ 0 & \text{otherwise.} \end{cases}$$

Denote $x^1 = x$ and $x^0 = \neg x$.
We will view F as a set of clauses and each clause as a set of literals; e.g. $F = \{\{x, \neg y\}, \{\neg x, y, z\}\}$ instead of $F = (x \vee \neg y) \wedge (\neg x \vee y \vee z)$

- *leaf node*:

DP: primal treewidth II

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- *leaf node*: $\text{sat}(t, \{x = a\}) = \begin{cases} 1 & \text{if } \{x^{1-a}\} \notin F \\ 0 & \text{otherwise} \end{cases}$
- *introduce node*:

DP: primal treewidth II

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- *leaf node*: $\mathsf{sat}(t, \{x = a\}) = \begin{cases} 1 & \text{if } \{x^{1-a}\} \notin F \\ 0 & \text{otherwise} \end{cases}$
- *introduce node*: $\gamma(t) = \gamma(t') \cup \{x\}$.

$$\mathsf{sat}(t, \{x = a\} \cup \{x_i = a_i\}_i) = \mathsf{sat}(t', \{x_i = a_i\}_i) \\ \wedge \nexists C \in F : C \subseteq \{x^{1-a}\} \cup \{x_i^{1-a_i}\}_i.$$

DP: primal treewidth III

- *forget node*:

DP: primal treewidth III

- *forget node*: $\gamma(t) = \gamma(t') \setminus \{x\}$.

$$\begin{aligned}\mathsf{sat}(t, \{x_i = a_i\}_i) = \mathsf{sat}(t', \{x = 0\} \cup \{x_i = a_i\}_i) \\ \vee\, \mathsf{sat}(t', \{x = 1\} \cup \{x_i = a_i\}_i).\end{aligned}$$

- *join node*:

DP: primal treewidth III

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- *forget node*: $\gamma(t) = \gamma(t') \setminus \{x\}$.

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- *join node*:

$$\begin{aligned}\mathsf{sat}(t, \{x_i = a_i\}_i) = {} & \mathsf{sat}(t_1, \{x_i = a_i\}_i) \\ & \wedge \mathsf{sat}(t_2, \{x_i = a_i\}_i).\end{aligned}$$

- Finally: F is satisfiable iff $\exists \tau : \gamma(r) \to \{0,1\}$ such that $\mathsf{sat}(r, \tau) = 1$
- Running time: $O^*(2^k)$, where k is the primal treewidth of F, assuming we are given a minimum width tree decomposition
- Also extends to computing the number of satisfying assignments

Direct Algorithms

Known treewidth based algorithms for SAT:

$k =$ primal tw	$k =$ dual tw	$k =$ incidence tw
$O^*(2^k)$	$O^*(2^k)$	$O^*(4^k)$

- It is still worth considering primal treewidth and dual treewidth.
- These algorithms all count the number of satisfying assignments.

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Constraint Satisfaction Problem

CSP

Input:	A set of variables X, a domain D, and a set of constraints C
Question:	Is there an assignment $\tau : X \to D$ satisfying all the constraints in C?

A constraint has a scope $S = (s_1, \ldots, s_r)$ with $s_i \in X, i \in \{1, \ldots, r\}$, and a constraint relation R consisting of r-tuples of values in D.
An assignment $\tau : X \to D$ satisfies a constraint $c = (S, R)$ if there exists a tuple $(d_1, \ldots, d_r)$ in R such that $\tau(s_i) = d_i$ for each $i \in \{1, \ldots, r\}$.

Bounded Treewidth for Constraint Satisfaction

- Primal, dual, and incidence graphs are defined similarly as for SAT.

Theorem 6 ([GSS02])

CSP is FPT for parameter primal treewidth if $|D| = O(1)$.

- What if domains are unbounded?

Unbounded domains

Theorem 7

CSP is W[1]-hard for parameter primal treewidth.

Unbounded domains

Theorem 7

CSP is W[1]-hard for parameter primal treewidth.

Proof Sketch.

Parameterized reduction from CLIQUE.
Let $(G = (V, E), k)$ be an instance of CLIQUE.
Take k variables $x_1, \dots, x_k$, each with domain V.
Add $\binom{k}{2}$ binary constraints $E_{i,j}$, $1 \le i < j \le k$.
A constraint $E_{i,j}$ has scope (x_i, x_j) and its constraint relation contains the tuple (u, v) if $uv \in E$.
The primal treewidth of this CSP instance is $k - 1$. $\square$

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Further Reading

- Chapter 7, *Treewidth* in [Cyg+15]
- Chapter 5, *Treewidth* in [FK10]
- Chapter 10, *Tree Decompositions of Graphs* in [Nie06]
- Chapter 10, *Treewidth and Dynamic Programming* in [DF13]
- Chapter 13, *Courcelle's Theorem* in [DF13]

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