GSOE9210 Engineering Decisions

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Mixed strategies

Outline

Mixed strategies

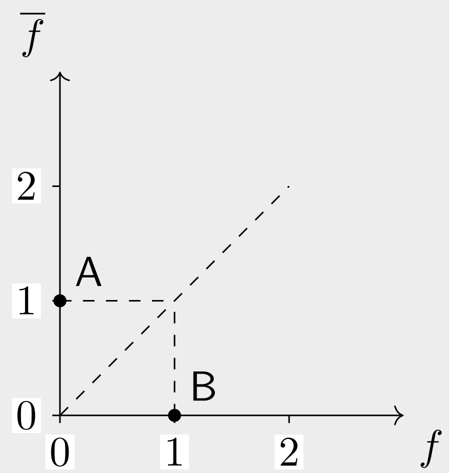
Consider the River problem described earlier:

- Action C is weakly dominated by B; disregard it
- Value and regret tables:

	f	$\overline{f}$
A	4	0
B	3	1

	f	$\overline{f}$
A	0	1
B	1	0

Regret plot:



- Original values: fuel saved;
  Regret: extra fuel used

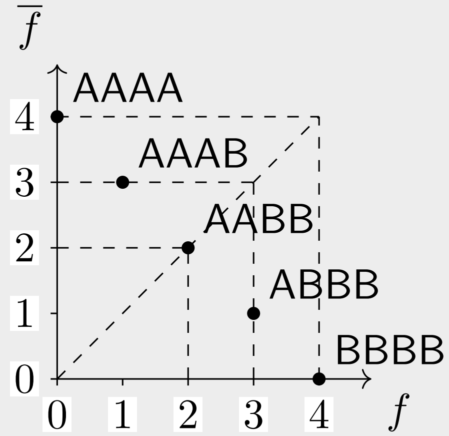
Mixed strategies

Example (Multi-decision strategies)

Suppose four packages have to be delivered urgently to C today. Each package is transported on a separate motor-boat.

	f	$\overline{f}$
A	0	1
B	1	0

	f	$\overline{f}$
AAAA	0	4
AAAB	1	3
AABB	2	2
ABBB	3	1
BBBB	4	0

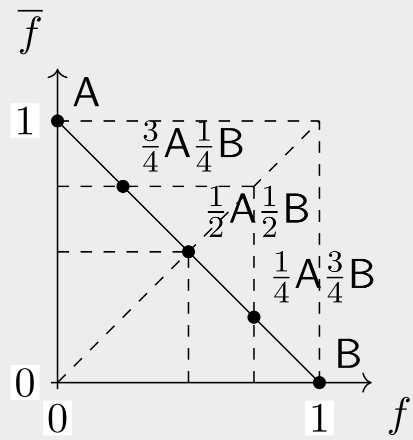


- Strategy AAAB: three trips via A and one via B;
  One extra litre used if f (due to $1 \times$ B) and three if $\overline{f}$ ($3 \times$ A)

Mixed strategies

Average over many trips (*i.e.*, per trip):

	f	$\overline{f}$
A	0	1
$\frac{3}{4}$A$\frac{1}{4}$B	$\frac{1}{4}$	$\frac{3}{4}$
$\frac{1}{2}$A$\frac{1}{2}$B	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{4}$A$\frac{3}{4}$B	$\frac{3}{4}$	$\frac{1}{4}$
B	1	0

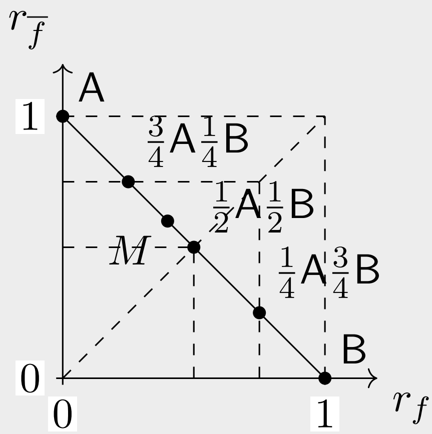


Definition (Mixed strategy)

A *mixed strategy* (or *mixture*) is a strategy in which the basic strategies are distributed proportionately. A strategy in which the entire proportion is from one basic strategy is called a *pure strategy*.

Mixed strategies

	f	$\overline{f}$
A	0	1
$\frac{3}{4}$A$\frac{1}{4}$B	$\frac{1}{4}$	$\frac{3}{4}$
$\frac{1}{2}$A$\frac{1}{2}$B	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{4}$A$\frac{3}{4}$B	$\frac{3}{4}$	$\frac{1}{4}$
B	1	0



- Mixtures of A and B lie on line segment AB
- Positioning of mixture M determined by *mixture parameter* μ_{A} $(0 \leqslant \mu_{\mathsf{A}} \leqslant 1)$
- *i.e.*, if $M = M(\mu_{\mathsf{A}})$ then M is μ_{A} of the way from B to A; *e.g.*, $\frac{3}{4}$A$\frac{1}{4}$B $= M(\frac{3}{4})$

Mixed strategies

In general:

- If $\mathcal{A} = \{a_1, \ldots, a_k\}$, then the mixed strategies are determined by the mixtures $(\mu_{a_1}, \ldots, \mu_{a_k})$ associated with the basic strategies
- The value of mixed strategy $M(\mu_{a_1}, \ldots, \mu_{a_k})$ in state $s \in \mathcal{S}$ is the average value of the basic strategies:

$$\begin{aligned} V(M, s) &= \mu_{a_1} v(a_1, s) + \cdots + \mu_{a_k} v(a_k, s) \\ &= \sum_{a \in \mathcal{A}} \mu_a v(a, s). \end{aligned}$$

where

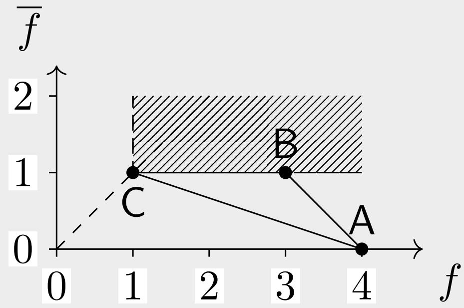
$$\sum_{a \in \mathcal{A}} \mu_a = 1$$

- Think of a mixture as many independent decisions in a single unknown state

River example

- Axes correspond to payoffs in each of the two states; *i.e.*, payoff v_1 in state $s_1 = f$ and v_2 in $s_2 = \overline{f}$
- Actions graphed below:

	f	$\overline{f}$
A	4	0
B	3	1
C	1	1



- Option C not a better response than B under any circumstances (*i.e.*, in any state)
- C worse than B in some cases and no better in all others; C can be *discarded*

Mixed strategies: mixture plots

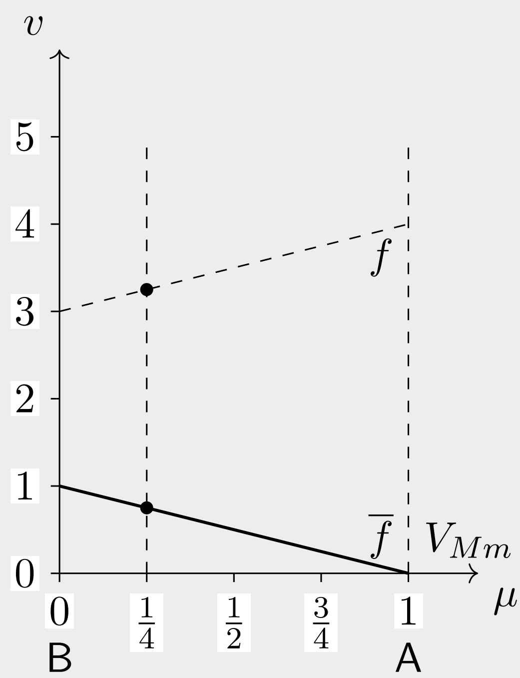
For the river problem with $\mu_{\text{A}} = \mu$:

	f	$\overline{f}$
A	4	0
B	3	1
M	$3\frac{1}{4}$	$\frac{3}{4}$

$$m_1 = 4\mu + 3(1-\mu) = 3 + \mu$$
$$m_2 = 0\mu + 1(1-\mu) = 1 - \mu$$

Exercise

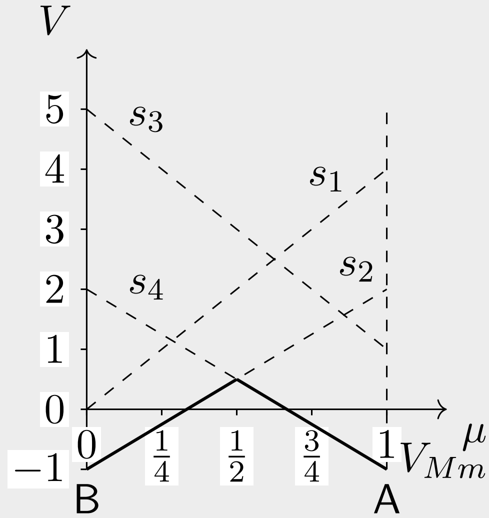
Which is the *Maximin* mixed strategy?



Mixed strategies: many states

Consider a problem with four states, two basic strategies, and mixtures, M, where $\mu_A = \mu$:

	s_1	s_2	s_3	s_4
A	4	2	1	-1
B	0	-1	5	2
$M(\mu)$	4μ	$3\mu - 1$	$5 - 4\mu$	$2 - 3\mu$



Maximin values for mixed strategies $M(\mu)$ lie on solid line.
Maximin mixed strategy M^* given by $\mu^* = \frac{1}{2}$ which maximises *Maximin* values; *i.e.*, $V_{Mm}(M^*) = \frac{1}{2}$.

Outline

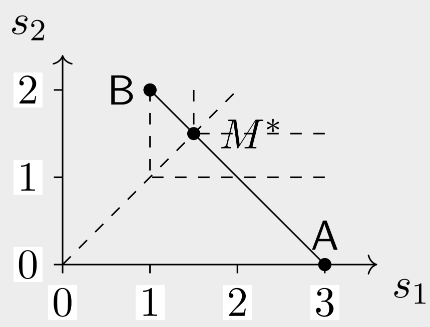
1 What are mixed strategies?

2 Calculations with mixtures

3 Mixing many strategies

Mixed strategies: *Maximin*

		s_1	s_2	V_{Mm}
μ	A	3	0	0
$1-\mu$	B	1	2	1
	$M(\mu)$	m_1	m_2	m
	M^*	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$



- Mixtures defined by *mixture parameter* μ $(0 \leqslant \mu \leqslant 1)$:
  $M(\mu) = (2\mu + 1, 2 - 2\mu)$; *i.e.*, $m_1 = 2\mu + 1$, $m_2 = 2 - 2\mu$
- Point M^* corresponds to mixture $M(\frac{1}{4}) = \frac{1}{4}\text{A}\frac{3}{4}\text{B}$

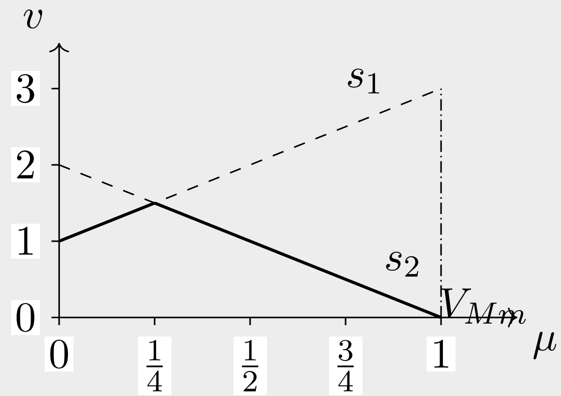
Exercise

Derive a general expression for a mixture $M(\mu)$ of two actions A and B.

Mixed strategies: mixture plot

Consider mixtures M, where $\mu_A = \mu$:

	s_1	s_2
A	3	0
B	1	2
$M(\mu)$	$2\mu + 1$	$2 - 2\mu$



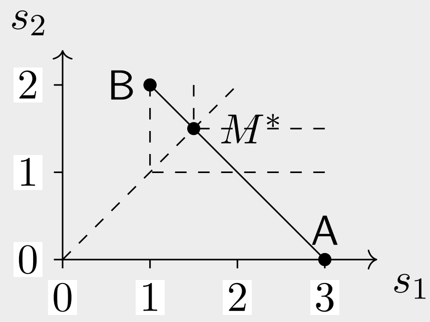
- *Maximin* values for mixed strategies $M(\mu)$ lie on solid line
- *Maximin* mixed strategy M^* is mixture that maximises *Maximin* value
- *Maximin* value maximised for $\mu^* = \frac{1}{4}$; *i.e.*, $V_{Mm}(M^*) = \frac{3}{2}$

Exercises

Verify algebraically the value of μ^* above.

Mixed strategies: *Maximin*

		s_1	s_2	V_{Mm}
μ	A	3	0	0
$1-\mu$	B	1	2	1
	$M(\mu)$	m_1	m_2	m
	M^*	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$



- Mixtures defined by *mixture parameter* μ $(0 \leqslant \mu \leqslant 1)$:
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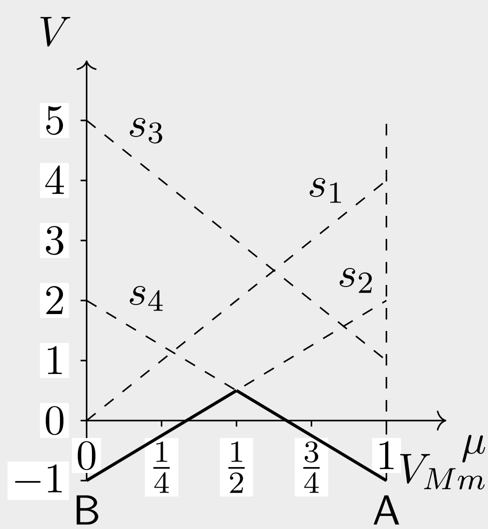
Exercise

Derive a general expression for a mixture $M(\mu)$ of two actions A and B.

Mixed strategies: many states

Consider a problem with four states, two basic strategies, and mixtures, M, where $\mu_A = \mu$:

	s_1	s_2	s_3	s_4
A	4	2	1	-1
B	0	-1	5	2
$M(\mu)$	4μ	$3\mu - 1$	$5 - 4\mu$	$2 - 3\mu$

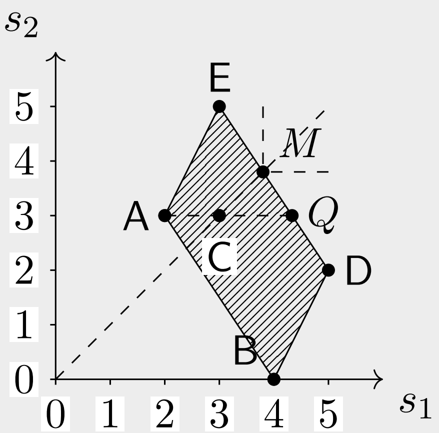


Maximin values for mixed strategies $M(\mu)$ lie on solid line.
Maximin mixed strategy M^* given by $\mu^* = \frac{1}{2}$ which maximises *Maximin* values; *i.e.*, $V_{Mm}(M^*) = \frac{1}{2}$.

Outline

Mixed strategies: many basic strategies

	s_1	s_2
A	2	3
B	4	0
C	3	3
D	5	2
E	3	5



- Can mix more than two strategies: *e.g.*, $\mathsf{C} = \mu_{\mathsf{A}}\mathsf{A}\mu_{\mathsf{E}}\mathsf{E}\mu_{\mathsf{D}}\mathsf{D}$
- Mixtures lie inside (or on the boundary of) the shaded region. Why?

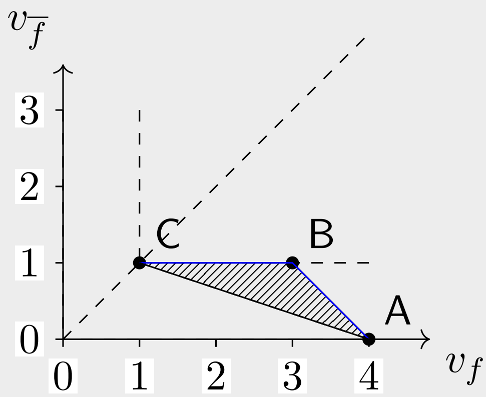
Exercise

Which is the *Maximin* mixed strategy? What is its value?

Mixed strategies

The River decision problem:

	f	$\overline{f}$
A	4	0
B	3	1
C	1	1

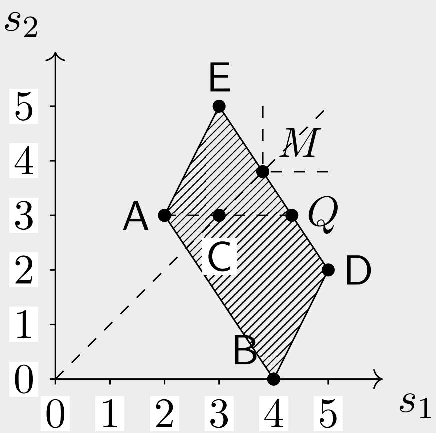


Exercises

- Are AC mixtures ever better than BC mixtures? AB mixtures? Others?
- Which mixtures are admissible (not dominated)?
- Determine the *Maximin* mixed strategy? What is its value?

Mixed strategies: many basic strategies

	s_1	s_2
A	2	3
B	4	0
C	3	3
D	5	2
E	3	5



- Can mix more than two strategies: *e.g.*, C $= \mu_{\mathsf{A}}\mathsf{A}\mu_{\mathsf{E}}\mathsf{E}\mu_{\mathsf{D}}\mathsf{D}$
- Mixtures lie inside (or on the boundary of) the shaded region. Why?

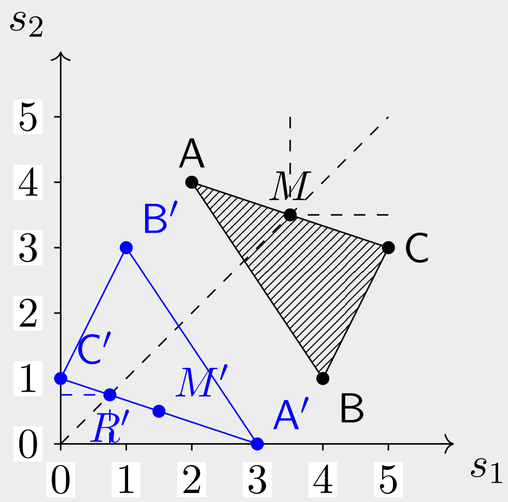
Exercise

Which is the *Maximin* mixed strategy? What is its value?

Mixed strategies: *miniMax Regret*

Consider the regret for the decision problem below:

	s_1	s_2
A	2	4
B	4	1
C	5	3
M	m_1	m_2



Note: *miniMax Regret* mixed action R' doesn't correspond to *Maximin* mixed action M.

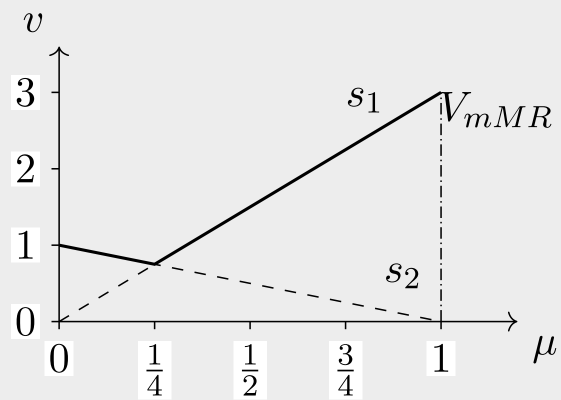
Exercise

Determine the *miniMax Regret* mixed strategy. What is its value?

Mixed strategies: mixture plot

Consider mixtures M, where $\mu_A = \mu$:

	s_1	s_2
A	3	0
C	0	1
M	3μ	$1-\mu$



- *miniMax Regret* values for mixed strategies $M(\mu)$ lie on solid line
- *miniMax Regret* mixed strategy M^* is mixture that minimises *miniMax Regret* value
- *miniMax Regret* value maximised for $\mu^* = \frac{1}{4}$; *i.e.*, $V_{mMR}(M^*) = \frac{3}{4}$

Exercises

Verify algebraically the value of μ^* above.

Generalised dominance

Definition (Strict dominance)

Strategy A *strictly dominates* B iff every outcome of A is more preferred than the corresponding outcome of B.

Definition (Weak dominance)

Strategy A *weakly dominates* B iff every outcome of A is no less preferred than the corresponding outcome of B, and some outcome is more preferred.

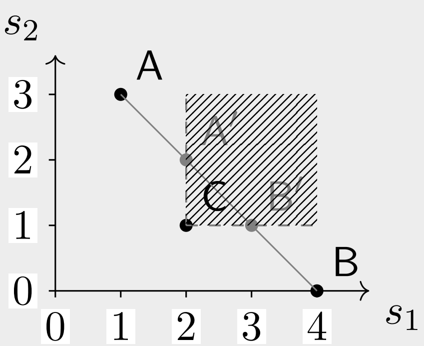
	s_1	s_2	s_3
A	3	4	2
B	4	4	3
C	5	6	3

Exercise

Which strategies in the decision table shown are dominated?

Mixed strategies: dominance

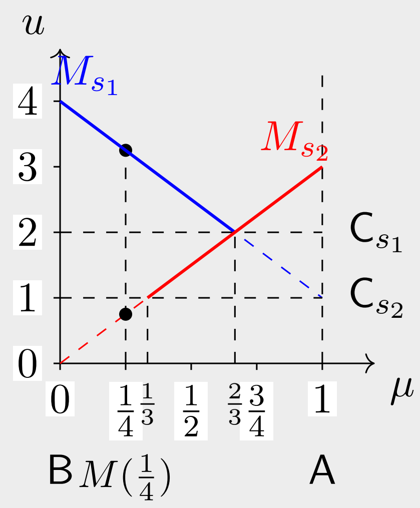
	s_1	s_2
A	1	3
B	4	0
~~C~~	~~2~~	~~1~~



- No pure strategies dominated by other pure strategies
- However, C is dominated by all mixed strategies on A′B′
- C isn't admissible among *mixed strategies*

Mixed strategies: dominance

	s_1	s_2
A	1	3
B	4	0
C	2	1
M	$4-3\mu$	3μ



Let $M_{\mathsf{AB}}(\mu) = \mu\mathsf{A} + (1-\mu)\mathsf{B}$; *i.e.*,

$$M(\mu) = (M_{s_1}(\mu), M_{s_2}(\mu))$$
$$= (4-3\mu, 3\mu)$$

For example,

$$M(\tfrac{1}{4}) = (3\tfrac{1}{4}, \tfrac{3}{4})$$

- Dominance requires: $4 - 3\mu \geqslant 2$; *i.e.*, $\mu \leqslant \frac{2}{3}$
- Similarly: $3\mu \geqslant 1$; *i.e.*, $\mu \geqslant \frac{1}{3}$.
- C dominated when *both* of the above hold: *i.e.*, when $\frac{1}{3} \leqslant \mu \leqslant \frac{2}{3}$

Summary: mixed strategies

- Mixed strategies as combinations of pure strategies
- Interpreting mixed strategies are repeated decisions about a single event
- Visualising and plotting mixtures: mixture plots
- Mixtures and dominance