What are mixed strategies?

Calculations with mixtures

Mixing many strategies
What are mixed strategies?

Consider the River problem described earlier:

- Action C is weakly dominated by B; disregard it.
- Value and regret tables:

<table>
<thead>
<tr>
<th></th>
<th>f</th>
<th>( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>f</th>
<th>( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

- Original values: fuel saved; Regret: extra fuel used

Regret plot:
**Mixed strategies**

**Example (Multi-decision strategies)**

Suppose four packages have to be delivered urgently to C today. Each package is transported on a separate motor-boat.

<table>
<thead>
<tr>
<th></th>
<th>$f$</th>
<th>$\bar{f}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAAA</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>AAAB</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>AABB</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>ABBB</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>BBBBB</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

- Strategy AAAB: three trips via A and one via B; One extra litre used if $f$ (due to $1 \times B$) and three if $\bar{f}$ ($3 \times A$)

**Definition (Mixed strategy)**

A *mixed strategy* (or *mixture*) is a strategy in which the basic strategies are distributed proportionately. A strategy in which the entire proportion is from one basic strategy is called a *pure strategy*. 
Mixed strategies

In general:

- If \( A = \{a_1, \ldots, a_k\} \), then the mixed strategies are determined by the mixtures \((\mu_{a_1}, \ldots, \mu_{a_k})\) associated with the basic strategies.

- The value of mixed strategy \( M(\mu_{a_1}, \ldots, \mu_{a_k}) \) in state \( s \in S \) is the average value of the basic strategies:

\[
V(M, s) = \mu_{a_1} v(a_1, s) + \cdots + \mu_{a_k} v(a_k, s) = \sum_{a \in A} \mu_a v(a, s).
\]

where

\[
\sum_{a \in A} \mu_a = 1
\]

- Think of a mixture as many independent decisions in a single unknown state.

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What are mixed strategies?

**River example**

- Axes correspond to payoffs in each of the two states; *i.e.*, payoff $v_1$ in state $s_1 = f$ and $v_2$ in $s_2 = \overline{f}$
- Actions graphed below:

<table>
<thead>
<tr>
<th></th>
<th>$f$</th>
<th>$\overline{f}$</th>
</tr>
</thead>
</table>
  A  | 4   | 0           |
  B  | 3   | 1           |
  C  | 1   | 1           |

- Option C not a better response than B under any circumstances (*i.e.*, in any state)
- C worse than B in some cases and no better in all others; C can be *discarded*

**Mixed strategies: mixture plots**

For the river problem with $\mu_A = \mu$:

<table>
<thead>
<tr>
<th></th>
<th>$f$</th>
<th>$\overline{f}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>M</td>
<td>$3\frac{1}{4}$</td>
<td>$3\frac{3}{4}$</td>
</tr>
</tbody>
</table>

$m_1 = 4\mu + 3(1 - \mu) = 3 + \mu$

$m_2 = 0\mu + 1(1 - \mu) = 1 - \mu$

**Exercise**

Which is the *Maximin* mixed strategy?
Mixed strategies: many states

Consider a problem with four states, two basic strategies, and mixtures, \( M \), where \( \mu_A = \mu \):

\[
\begin{array}{c|cccc}
   & s_1 & s_2 & s_3 & s_4 \\
\hline
A & 4 & 2 & 1 & -1 \\
B & 0 & -1 & 5 & 2 \\
\end{array}
\]

\[
M(\mu) = 4\mu \quad 3\mu - 1 \quad 5 - 4\mu \quad 2 - 3\mu
\]

Maximin values for mixed strategies \( M(\mu) \) lie on solid line. Maximin mixed strategy \( M^* \) given by \( \mu^* = \frac{1}{2} \) which maximises Maximin values; i.e., \( V_{Mm}(M^*) = \frac{1}{2} \).
Mixed strategies: Maximin

\[
\begin{array}{c|cc|c|c}
\mu & s_1 & s_2 & V_{Mm} & s_2 \\
1 - \mu & A & 3 & 0 & 0 \\
& B & 1 & 2 & 1 \\
\hline
M(\mu) & m_1 & m_2 & m & m \\
M^* & 3/2 & 3/2 & 3/2 & 3/2 \\
\end{array}
\]

- Mixtures defined by mixture parameter \( \mu \) (0 ≤ \( \mu \) ≤ 1):
  \[ M(\mu) = (2\mu + 1, 2 - 2\mu); \text{ i.e., } m_1 = 2\mu + 1, m_2 = 2 - 2\mu \]
- Point \( M^* \) corresponds to mixture \( M(\frac{1}{4}) = \frac{1}{4}A\frac{3}{4}B \)

**Exercise**

Derive a general expression for a mixture \( M(\mu) \) of two actions A and B.

Mixed strategies: mixture plot

Consider mixtures \( M \), where \( \mu_A = \mu \):

\[
\begin{array}{c|cc}
\ & s_1 & s_2 \\
A & 3 & 0 \\
B & 1 & 2 \\
\hline
M(\mu) & 2\mu + 1 & 2 - 2\mu \\
\end{array}
\]

- \textit{Maximin} values for mixed strategies \( M(\mu) \) lie on solid line
- \textit{Maximin} mixed strategy \( M^* \) is mixture that maximises \textit{Maximin} value
- \textit{Maximin} value maximised for \( \mu^* = \frac{1}{4} \); i.e., \( V_{Mm}(M^*) = \frac{3}{2} \)

**Exercises**

Verify algebraically the value of \( \mu^* \) above.
Mixed strategies: **Maximin**

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>A</th>
<th>1 - $\mu$</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$s_1$</td>
<td>$s_2$</td>
<td>$V_{Mm}$</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$M(\mu)$</td>
<td>$m_1$</td>
<td>$m_2$</td>
<td>$m$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2}$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{3}{2}$</td>
</tr>
</tbody>
</table>

- Mixtures defined by *mixture parameter* $\mu$ ($0 \leq \mu \leq 1$): $M(\mu) = (2\mu + 1, 2 - 2\mu)$; i.e., $m_1 = 2\mu + 1$, $m_2 = 2 - 2\mu$
- Point $M^*$ corresponds to mixture $M\left(\frac{1}{4}\right) = \frac{1}{4}A\frac{3}{4}B$

**Exercise**

Derive a general expression for a mixture $M(\mu)$ of two actions A and B.

Mixed strategies: many states

Consider a problem with four states, two basic strategies, and mixtures, $M$, where $\mu_A = \mu$:

<table>
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<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>-1</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>$M(\mu)$</td>
<td>$4\mu$</td>
<td>$3\mu - 1$</td>
<td>$5 - 4\mu$</td>
<td>$2 - 3\mu$</td>
</tr>
</tbody>
</table>

**Maximin** values for mixed strategies $M(\mu)$ lie on solid line.

**Maximin** mixed strategy $M^*$ given by $\mu^* = \frac{1}{2}$ which maximises **Maximin** values; i.e., $V_{Mm}(M^*) = \frac{1}{2}$. 

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Outline

1. What are mixed strategies?
2. Calculations with mixtures
3. Mixing many strategies

Mixed strategies: many basic strategies

<table>
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<tbody>
<tr>
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<td>3</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>D</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>E</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

- Can mix more than two strategies: e.g., $C = \mu_A A \mu_E E \mu_D D$
- Mixtures lie inside (or on the boundary of) the shaded region. Why?

Exercise

Which is the *Maximin* mixed strategy? What is its value?
Mixed strategies

The River decision problem:

<table>
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<th>( \bar{f} )</th>
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<tbody>
<tr>
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<td>0</td>
</tr>
<tr>
<td>B</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Exercises

- Are AC mixtures ever better than BC mixtures? AB mixtures? Others?
- Which mixtures are admissible (not dominated)?
- Determine the *Maximin* mixed strategy? What is its value?

Mixed strategies: many basic strategies

<table>
<thead>
<tr>
<th>( s_1 )</th>
<th>( s_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
</tr>
<tr>
<td>D</td>
<td>5</td>
</tr>
<tr>
<td>E</td>
<td>3</td>
</tr>
</tbody>
</table>

- Can mix more than two strategies: e.g., \( C = \mu_A A + \mu_E E + \mu_D D \)
- Mixtures lie inside (or on the boundary of) the shaded region. Why?

Exercise

Which is the *Maximin* mixed strategy? What is its value?
Mixed strategies: *miniMax Regret*

Consider the regret for the decision problem below:

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>$M$</td>
<td>$m_1$</td>
<td>$m_2$</td>
</tr>
</tbody>
</table>

Note: *miniMax Regret* mixed action $R'$ doesn't correspond to *Maximin* mixed action $M$.

**Exercise**

Determine the *miniMax Regret* mixed strategy. What is its value?

---

Mixed strategies: mixture plot

Consider mixtures $M$, where $\mu_A = \mu$:

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$M$</td>
<td>$3\mu$</td>
<td>$1 - \mu$</td>
</tr>
</tbody>
</table>

- *miniMax Regret* values for mixed strategies $M(\mu)$ lie on solid line
- *miniMax Regret* mixed strategy $M^*$ is mixture that minimises *miniMax Regret* value
- *miniMax Regret* value maximised for $\mu^* = \frac{1}{4}$; i.e., $V_{mMR}(M^*) = \frac{3}{4}$

**Exercises**

Verify algebraically the value of $\mu^*$ above.
Mixing many strategies

Generalised dominance

**Definition (Strict dominance)**
Strategy $A$ strictly dominates $B$ iff every outcome of $A$ is more preferred than the corresponding outcome of $B$.

**Definition (Weak dominance)**
Strategy $A$ weakly dominates $B$ iff every outcome of $A$ is no less preferred than the corresponding outcome of $B$, and some outcome is more preferred.

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>3</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>$B$</td>
<td>4</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>$C$</td>
<td>5</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

**Exercise**
Which strategies in the decision table shown are dominated?

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Mixed strategies: dominance

- No pure strategies dominated by other pure strategies
- However, $C$ is dominated by all mixed strategies on $A'B'$
- $C$ isn’t admissible among mixed strategies
Mixed strategies: dominance

Let \( M_{AB}(\mu) = \mu A + (1 - \mu)B; \) i.e.,

\[
M(\mu) = (M_{s1}(\mu), M_{s2}(\mu)) = (4 - 3\mu, 3\mu)
\]

For example,

\[
M\left(\frac{1}{4}\right) = (3\frac{1}{4}, 3\frac{1}{4})
\]

- Dominance requires: \( 4 - 3\mu \geq 2; \) i.e., \( \mu \leq \frac{2}{3} \)
- Similarly: \( 3\mu \geq 1; \) i.e., \( \mu \geq \frac{1}{3} \).
- C dominated when both of the above hold: i.e., when \( \frac{1}{3} \leq \mu \leq \frac{2}{3} \).

Summary: mixed strategies

- Mixed strategies as combinations of pure strategies
- Interpreting mixed strategies are repeated decisions about a single event
- Visualising and plotting mixtures: mixture plots
- Mixtures and dominance