Risk attitudes

1 Utility
- Bets and odds
- Expected monetary value
- Risk attitudes
You have $1000. Would you risk it to play ‘double or nothing’ on the toss of a fair coin? i.e., to win $2000 on heads, and $0 on tails?

- Measured in dollars, \( v_S(x) = x \), the two have equal Bayes value; i.e.,
  \[ v_S(1000) = 1000 = V_B\left(\frac{1}{2} : 2000 \mid \frac{1}{2} : 0\right) \]
- most people prefer a certain $1000 over an even chance at $2000 or $0; i.e., prefer $1000 to \( \frac{1}{2} : 2000 \mid \frac{1}{2} : 0 \)
- what value function, \( u \), on monetary outcomes would satisfy:
  \[ V_B([1000]) = u(1000) > V_B\left(\frac{1}{2} : 2000 \mid \frac{1}{2} : 0\right) \]
Money bets and odds

**Example (Betting)**

Alice has $4 to bet on the toss of a fair coin to win $10 on heads.

Should Alice gamble?

\[
\begin{align*}
\ell_G &= [\frac{1}{2} : $10, \frac{1}{2} : $0] \\
\ell_{\overline{G}} &= [$4]
\end{align*}
\]

**Definition (Expected monetary value)**

The expected monetary value (EMV) of a lottery, denoted \( V_\$ \), is the Bayes value of the lottery when outcomes are valued in $ (i.e., \( v = v_\$ \)).

\[
V_\$(\ell_{\overline{G}}) = 4 \\
V_\$(\ell_G) = \frac{1}{2}v_\$(h) + \frac{1}{2}v_\$(t) \\
= \frac{1}{2}(10) + \frac{1}{2}(0) \\
= 5
\]

The bet Alice was offered was fair—indeed ‘favourable’—for Alice; i.e., \( V_\$(\ell_G) > V_\$(\ell_{\overline{G}}) \)

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Expected monetary value

**Definition (Fair bet)**

A two-way gamble is represented by a binary lottery. A bet is fair for an agent if its expected monetary value for the corresponding lottery is no less than the value of not gambling; i.e.,

\[
V_\$(\ell_G) = E(v_\$) \geq V_\$(\ell_{\overline{G}})
\]

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Utility  
Expected monetary value

Bets, stakes, and odds

Example (The races)

Alice is at the races and she’s offered odds of ‘13 to 2’ (13 : 2) on a horse by a bookmaker; i.e., for every $2 she puts in (her stake), the bookmaker puts in $13, and the winner takes the entire pool ($15 = $13 + $2).

Should Alice gamble? i.e., is the bet favourable for Alice?

Definition (Favourable bet)

A bet is favourable to an agent if the value of the corresponding lottery for the agent is greater than that of not gambling. It is unfavourable if it is neither fair nor favourable.

Theorem (Fair bets)

Let \( a \) be agent A’s stake and \( b \) be B’s stake in a bet in which \( p \) is A’s probability of winning. The bet is fair iff:

\[
\frac{a}{b} = \frac{p}{1 - p}
\]
Bets: belief

- Suppose Alice believes that her horse has a 20% chance of winning.
- Then:

\[
V_\$ (\ell_G) = \frac{1}{5} (15) + \frac{4}{5} (0) = \$3
\]

\[
V_\$ (\ell_G) = \$2.
\]

- Hence bet is favourable according to Alice based on her beliefs about her chance of winning.

Exercises

- Prove the theorem on fair bets.
- For what probabilities of winning should Alice bet on her horse?

Working example

Example

A bookmaker (B) offers Alice (A) odds ‘4 to 1’ (4 : 1) on her team—a strong underdog—to win a football match. Alice has $10 to bet on her team.

- The ‘bookie’ puts up $4 for every $1 Alice bets, so the bookie has to put $40 into the pool to match Alice’s $10
- Alice’s outcomes: balance of $50 or $0, depending on whether her team wins or loses
- a bet is fair overall if it is not unfavourable to both parties involved; i.e., if both parties expect to get back what they put in
Fair bets

The decision tree for the two-way bet:

- **G** means Alice’s agrees to gamble, and \( p_A \) is the probability that Alice wins (\( p_A + p_B = 1 \))

Fair odds (in $):

\[
p_A(50) + (1 - p_A)(0) \geq 10
\]

\[i.e. \quad p_A \geq \frac{10}{40+10} = \frac{1}{5}\]

In general, a bet is fair for A if:

\[
p_A \geq \frac{x_A}{x_A + x_B}
\]

where \( x_A \) is A’s stake ($10)

\( x_B \) is B’s stake ($40).

Utility of bets

- Bet would be fair if Alice believes chances of her team winning exceed 1 in 5 . . . Suppose Alice needs $10 to buy dinner; should Alice gamble?

- Suppose Alice’s preferences are: I’ll gamble (risk going hungry) only if I believe my team’s chances are at least even (i.e., greater than 1 in 2)

- That is, Alice indifferent between certain $10 and \([\frac{1}{2} : 50|\frac{1}{2} : 0]\):

\[
u(10) = U([\frac{1}{2} : 50|\frac{1}{2} : 0]) = E(u)
\]

\[= V_B([\frac{1}{2} : 50|\frac{1}{2} : 0]) \quad \text{using } u \text{ rather than } v_S
\]

\[= \frac{1}{2} u(50) + \frac{1}{2} u(0)\]

- What does \( u \) look like?
Utility for money

Fix utility scale:

\[ u(\$0) = 0 \]
\[ u(\$50) = 1 \]

Possible gambles lie on diagonal:

\[ U\left(\left[\frac{1}{2} \right] \cdot \text{\$50} \left| \left[\frac{1}{2} \right] \cdot \text{\$0}\right]\right) = \frac{1}{2} u(\$50) + \frac{1}{2} u(\$0) = \frac{1}{2} \]
\[ U\left(\left[p \cdot \text{\$50} \left| (1-p) \cdot \text{\$0}\right]\right]\right) = p \]

On Alice’s utility scale the monetary outcomes are arranged as follows:

<table>
<thead>
<tr>
<th>$0$</th>
<th>$\frac{1}{5}$</th>
<th>$\frac{1}{2}$</th>
<th>$\frac{9}{10}$</th>
<th>$1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$</td>
<td>$u$</td>
<td>$u$</td>
<td>$u$</td>
<td>$u$</td>
</tr>
</tbody>
</table>

Utility values should increase with increasing money.

What properties do typical utility functions for money have?
Functions on ordered sets

Definition (Monotonic increasing function)

A real-valued function $f : \mathbb{R} \rightarrow \mathbb{R}$ is *monotonically increasing*, or *non-decreasing*, iff for any $x, y \in \mathbb{R}$, if $x \geq y$, then $f(x) \geq f(y)$.

Examples: the following are non-decreasing functions on $\mathbb{R}$: $f(x) = \frac{1}{10} x$, $f(x) = x$, $f(x) = c$, for any fixed $c \in \mathbb{R}$.

Exercise

Does this imply the converse; i.e., if $f(x) \geq f(y)$, then $x \geq y$?

Strictly increasing functions

Definition (Strictly increasing function)

A real-valued function $f : \mathbb{R} \rightarrow \mathbb{R}$ is *strictly increasing* iff for any $x, y \in \mathbb{R}$, if $x > y$, then $f(x) > f(y)$.

Examples: $f(x) = \frac{1}{10} x$, $f(x) = x$, $f(x) = 3x + 2$, $f(x) = x^2$, $f(x) = \log_2 x$.

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### Utility for money

#### Definition (Certainty equivalent)
An agent’s *certainty equivalent* for a lottery is the value $x_c$ for which the agent would be indifferent between it and the lottery; i.e., $u(x_c) = U(\ell)$.

#### Definition (Risk premium)
The *risk premium* of an agent for lottery $\ell$ is the difference between the EMV of the lottery and the certainty equivalent: $V_S(\ell) - x_c$.

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**Example (Alice and Bob)**

Alice and her twin, Bob, have $10 each and they are offered, separately, 4 to 1 odds on a team in two different football matches (e.g., home and away). They believe the team has a 2 in 5 chance of winning each match.

### Repeated trials

#### Example (Alice and Bob)

Alice and her twin, Bob, have $10 each and they are offered, separately, 4 to 1 odds on a team in two different football matches (e.g., home and away). They believe the team has a 2 in 5 chance of winning each match.

**Should Alice bet?**

In terms of the individual outcomes of Alice and Bob:

$$\ell_{AB} = \left[ \begin{array}{c} \frac{9}{25} : (0, 0) \mid \frac{6}{25} : (0, 50) \mid \frac{6}{25} : (50, 0) \mid \frac{4}{25} : (50, 50) \end{array} \right]$$

If Alice and Bob share the risk/gain then:

$$(x, y) \sim $ \left( \frac{x + y}{2} \right) \quad i.e. \quad u_A(x, y) = u_A \left( \frac{x + y}{2} \right)$$

So for Alice:

$$\ell_A = \left[ \begin{array}{c} \frac{9}{25} : 0 \mid \frac{6}{25} : 25 \mid \frac{6}{25} : 25 \mid \frac{4}{25} : 50 \end{array} \right]$$

$$= \left[ \begin{array}{c} \frac{9}{25} : 0 \mid \frac{12}{25} : 25 \mid \frac{4}{25} : 50 \end{array} \right]$$
Where does $\ell_A$ fit in in the scheme of things?

$$\ell_A = \left[ \frac{9}{25} : 0 \right| \frac{12}{25} : 25 \right| \frac{4}{25} : 50 \right]$$

$V_S(\ell_A) = \frac{12}{25}(25) + \frac{4}{25}(50) = 20$

$U_A(\ell_A) = \frac{9}{25}(0) + \frac{12}{25}u_A(25) + \frac{4}{25}(1) = 0 + \frac{12}{25}\left(\frac{9}{10}\right) + \frac{4}{25}\left(\frac{37}{10}\right) > \frac{4}{25}\left(\frac{35}{10}\right) = \frac{14}{25} > \frac{1}{2} = u_A(10)$

Alice should bet, sharing the risk and the winnings!

- The individual bets are favourable for both Alice and Bob
- Despite this neither Alice nor Bob would take their respective individual bets
- However, they should bet together over multiple bets/trials
Risk attitudes

Definition (Risk attitudes)
An agent is:
- **risk averse** iff its certainty equivalent is less than the lottery’s expected value; i.e., it values the lottery to be worth less than the expected value.
- **risk seeking (risk prone)** iff its certainty equivalent is greater than the lottery’s expected value.
- **risk-neutral** otherwise.

Exercises
- What is Alice’s certainty equivalent for the lottery with Bob?
- The risk premium in what range if the agent is: risk averse? risk seeking? risk neutral?

More generally:

Definition (Risk averse)
An agent is **risk averse** if its utility function is concave down.

Definition (Risk seeking)
An agent is **risk seeking** if its utility function is concave up (convex).

Definition (Risk neutral)
An agent is **risk neutral** if its utility function both concave down and up; i.e., linear.
**Definition (Concave and convex)**

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is **concave down** in the interval $[a, b]$ if for all $x, y \in [a, b]$, and all $\lambda \in [0, 1]$, $f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$, and **concave up** (or **convex**) if $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$.

![Concave and convex functions diagram](image)

where $\nu = \lambda x + (1 - \lambda)y$

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**Summary: Introduction to utility**

- Not all quantities (e.g., $\$$) accurately represent preference over outcomes
- Expected values on these quantities may not accurately represent preference
- Measure preference in terms of utility; agent must calibrate utilities against uncertain outcomes (lotteries)
- An agent’s utility is personal/subjective; *i.e.*, particular to him. Different agents may have different utilities for the same ‘outcome’
- Utility functions are non-decreasing; this means that over many trials Bayes utilities approach expected values
- The shape of an agent’s utility curve/function determines its risk attitude