Exercise sheet 8 – Solutions

COMP6741: Parameterized and Exact Computation

Serge Gaspers

Semester 2, 2017

Exercise 1. Recall that a *k-coloring* of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, ..., k\}$ assigning colors to V such that no two adjacent vertices receive the same color.

COLORING	
Input:	Graph G, integer k
Question:	Does G have a k-coloring?

- Suppose A is an algorithm solving COLORING in $O(f(n))$ time, $n = |V|$, where f is non-decreasing. Design a $O^*(f(n))$ time algorithm B, which, for an input graph G, finds a coloring of G with a smallest number of colors.

Solution sketch.

1. First, compute the smallest number of colors needed to color G
   - For $k = 1$ to n, execute algorithm A for the instance (G, k), and stop when encountering the first Yes-instance.
     (Alternatively, use binary search to find the smallest k for which (G, k) is a Yes-instance)
2. Now, compute an actual k-coloring using the following ideas
   - Select two non-adjacent vertices u and v, and check whether G as a k-coloring where u and v receive distinct colors.
     This can be done by adding an edge between u and v, and using algorithm A.
     If there is such a k-coloring, add the edge uv, and continue with two other distinct vertices.
     If not, then u and v must receive the same color, and we merge them into a single vertex, and continue by picking two new non-adjacent vertices
   - A complete graph on ℓ vertices needs ℓ colors.

Exercise 2. Recall that a graph $G = (V, E)$ is *bipartite* if G has a 2-coloring. A *matching* in a graph $G = (V, E)$ is a set of edges $M \subseteq E$ such that no two edges of M have an end-point in common. The matching M in G is *perfect* if every vertex of G is contained in an edge of M.

#BIPARTITE PERFECT MATCHINGS	
Input:	Bipartite graph $G = (V, E)$
Output:	The number of perfect matchings in G

1. Design an algorithm for #BIPARTITE PERFECT MATCHINGS with running time $O^*\left(\left(\frac{n}{2}\right)!\right)$, where $n = |V|$.
2. Design a polynomial-space $O^*(2^{n/2})$-time inclusion-exclusion algorithm for #BIPARTITE PERFECT MATCHINGS.

Solution sketch.

1. Let (X, Y) be a bipartition of V such that X and Y are independent sets If $|X| \neq |Y|$, then return 0. Denote $X = \{x_1, \ldots, x_{n/2}\}$ and $Y = \{y_1, \ldots, y_{n/2}\}$. For each permutation $\pi = (y_{\pi(1)}, \ldots, y_{\pi(n/2)})$ of Y,

$$\{x_i y_{\pi(i)} : 1 \leq i \leq n/2\}$$

is a perfect matching iff $x_i y_{\pi(i)} \in E$ for each $i \in \{1, \ldots, n/2\}$.

2. U: contains each set of $n/2$ edges $\{e_1, \ldots, e_{n/2}\}$ such that $x_i \in e_i$. For each $v \in Y$, $A_v = \{S \in U : v \in \bigcup S\}$. The number of perfect matchings is

$$\begin{aligned}
\left| \bigcap_{v \in Y} A_v \right| &= \sum_{S \subseteq Y} (-1)^{|S|} \left| \bigcap_{v \in S} \overline{A_v} \right| \\
&= \sum_{S \subseteq Y} (-1)^{|S|} \prod_{i=1}^{n/2} |N(x_i) \setminus S|.
\end{aligned}$$