## Solution to COMP9334 Revision Questions for Week 5

### Question 1

First note that one counter is not sufficient to serve all the customers. If we consider all the customers together, each customer carries on average 20.5 items, which takes $\frac{20.5}{15}$ to complete. Since the customer arrival rate is 1, the utilisation will be above 1 if only one counter is used.

Let us refer to the two counters as Counter 1 and Counter 2. Let us assume that Counter 1 serves customers with $x$ items or less and Counter 2 serves customers with more than $x$ items, where $1 \leq x \leq 39$.

Let $\lambda(=1)$ denote the overall arrival rate and $\mu = 15$ be the service rate of each counter.

Let us consider Counter 1 first. Since only customers with $x$ items or less go to Counter 1, the arrival rate at Counter 1 is $\lambda\frac{x}{40}$. The customers arriving at Counter 1 bring with them 1, 2, ..., $x$ items uniformly distributed. Let $S_1$ denote the service time at Counter 1. We have

$$E[S_1] = \sum_{i=1}^{x} \frac{i}{\mu}\frac{1}{x} \tag{1}$$

$$E[S_1^2] = \sum_{i=1}^{x} \left(\frac{i}{\mu}\right)^2 \frac{1}{x} \tag{2}$$

Let $\rho_1 = \lambda\frac{x}{40}E[S_1]$, by the P-K formula, the mean waiting time at Counter 1 is

$$W_1 = \begin{cases} \lambda\frac{x}{40}\frac{E[S_1^2]}{2(1-\rho_1)} & \text{if } \rho_1 < 1 \\ \infty & \text{if } \rho_1 \geq 1 \end{cases} \tag{3}$$

Similarly, the arrival rate to Counter 2 is $\lambda_2 = \lambda\frac{40-x}{40}$. Let $S_2$ denote the service time at Counter 2, then

$$E[S_2] = \sum_{i=x+1}^{40} \frac{i}{\mu}\frac{1}{(40-x)} \tag{4}$$

$$E[S_2^2] = \sum_{i=x+1}^{40} \left(\frac{i}{\mu}\right)^2 \frac{1}{(40-x)} \tag{5}$$

Let $\rho_2 = \lambda\frac{40-x}{40}E[S_2]$, by the P-K formula, the mean waiting time at Counter 2 is

$$W_2 = \begin{cases} \lambda\frac{40-x}{40}\frac{E[S_2^2]}{2(1-\rho_2)} & \text{if } \rho_2 < 1 \\ \infty & \text{if } \rho_2 \geq 1 \end{cases} \tag{6}$$

The mean waiting time of the customers is

$$W = \frac{x}{40}W_1 + \frac{40-x}{40}W_2 \tag{7}$$

Note that $W$ is a function of $x$. We write a computer program (Matlab file *week05_q1.m*) to calculate how $W$ varies with $x$. Figure 1 shows how $W$ varies with $x$. It can be seem that the minimum value of $W$ is achieved at $x = 28$.

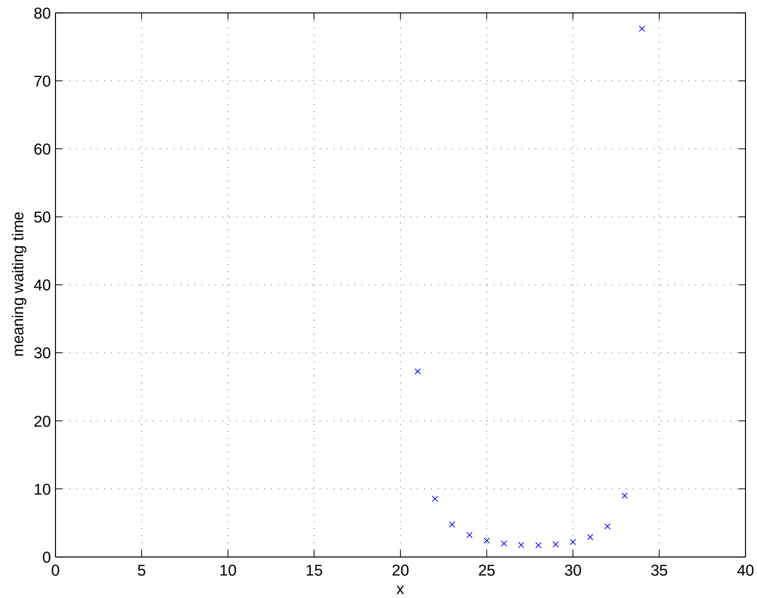


Figure 1: For Question 1

**Question 2**

Let us first compute the second moment of each customer type. Let us use $C_a$, $\sigma_a$, $E[S_a]$ and $E[S_a^2]$ to denote, respectively, the coefficient of variation, standard deviation, mean and second moment of the service time of customer of type $a$. Recall that the coefficient of variation of a random variable is its standard deviation divided by mean, i.e $C_a = \frac{\sigma_a}{E[S_a]}$. By using the relation:

$$E[S_a^2] = E[S_a]^2 + \sigma_a^2, \tag{8}$$

it can be showed that

$$E[S_a^2] = E[S_a]^2(1 + C_a^2) \tag{9}$$

Since we know $E[S_a] = 0.1$ and $C_a = 1.5$, we can compute $E[S_a^2]$ using the above equation.

Similarly, let $C_b$, $\sigma_b$, $E[S_b]$ and $E[S_b^2]$ to denote, respectively, the coefficient of variation, standard deviation, mean and second moment of the service time of customer of type $b$. We have

$$E[S_b^2] = E[S_b]^2(1 + C_b^2) \tag{10}$$

With $E[S_b] = 0.08$ and $C_b = 1.2$, we can compute $E[S_b^2]$ using the above equation.

Requests of type a and b have equal priorities

This is an M/G/1 queue without priority. The arrival rate is 10 requests per second ($= \lambda$) . Since 30% of the request are type a and the remaining are type b, we have the mean service time $E[S]$ and second moment $E[S^2]$ of the aggregate are, respectively,

$$\begin{aligned} E[S] &= 0.3E[S_a] + 0.7E[S_b] && (11) \\ E[S^2] &= 0.3E[S_a^2] + 0.7E[S_b^2] && (12) \end{aligned}$$

The mean response time is therefore $E[S] + \frac{\lambda E[S^2]}{2(1-\rho)}$ where $\rho = \lambda E[S]$.

Requests of type b have non-preemptive priority over type a

Let

$$\begin{aligned} R &= \frac{1}{2}(0.3\lambda E[S_a^2] + 0.7\lambda E[S_b^2]) && (13) \\ \rho_a &= 0.3\lambda E[S_a] && (14) \\ \rho_b &= 0.7\lambda E[S_b] && (15) \end{aligned}$$

Response time of type b is

$$E[S_b] + \frac{R}{1-\rho_b} \tag{16}$$

Response time of type a is

$$E[S_a] + \frac{R}{(1-\rho_b)(1-\rho_a-\rho_b)} \tag{17}$$

Requests of type b have preemptive priority over type a

Let

$$\begin{aligned} R_b &= \frac{1}{2}(0.7\lambda E[S_b^2]) && (18) \\ R_a &= \frac{1}{2}(0.3\lambda E[S_a^2] + 0.7\lambda E[S_b^2]) && (19) \\ \rho_a &= 0.3\lambda E[S_a] && (20) \\ \rho_b &= 0.7\lambda E[S_b] && (21) \end{aligned}$$

Response time of type b is

$$E[S_b] + \frac{R_b}{1-\rho_b} \tag{22}$$

Response time of type a is

$$E[S_a]\frac{1}{1-\rho_b} + \frac{R_a}{(1-\rho_b)(1-\rho_a-\rho_b)} \tag{23}$$

The numerical answers are summarised in the following table.

| Mean response time | Part 1 | Part 2 | Part 3 |
|---|---|---|---|
| type a | 0.8246 | 1.7787 | 1.9059 |
| type b | 0.8246 | 0.3150 | 0.2042 |

Observe that the response time for type b customers have become better because it has a higher priority, this is of course at the expense of type a customers which have a lower priority.

**Question 3**

The system behaves as an M/G/1 queueing system.

Since there are 10 sessions each generating Poisson traffic at a rate of 150 packets/minute, the packet arrival rate to the communication line is 1500 packets/minute or 25 packets/s ($= \lambda$).

With a transmission rate of 50 kbits/s, a 100-bit packet requires a transmission time (= service time in queueing theory terminology) 0.002s and a 1000-bit packet requires a transmission time of 0.02s. (Recall that transmission time is packet size divided by transmission rate.)

Given that 10% of the packets are 100 bits long and the rest are 1000 bits long, the mean service time $E[S]$ (where $S$ denotes the service time random variable)

$$E[S] = 0.1 * 0.002 + 0.9 * 0.02 = 0.0182s \tag{24}$$

and the second moment of the service time is

$$E[S^2] = 0.1 * 0.002^2 + 0.9 * 0.02^2 = 3.6040 \times 10^{-4} s^2 \tag{25}$$

The mean waiting time $W$, according to the P-K formula, which applies to M/G/1 queueing system, is

$$W = \frac{\lambda E[S^2]}{2(1 - \lambda E[S])} = 8.3ms \tag{26}$$

By Little's Law, the mean queue length is given by the product of the throughout of the queue and the mean waiting time,

$$\lambda * W = 0.21 \ \ packets \tag{27}$$