# 11. Kernel Lower Bounds
# COMP6741: Parameterized and Exact Computation

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# Outline

1. Reminder
2. Further Examples of Kernels
   - Kernel for Hamiltonian Cycle
   - Kernel for Edge Clique Cover
3. Frequently Arising Issues
4. Kernel Lower Bounds
   - Compositions
   - Polynomial Parameter Transformations
5. Further Reading

# Outline

# Kernelization

**Definition 1**

A kernelization (kernel) for a parameterized problem $\Pi$ is a **polynomial time** algorithm, which, for any instance $I$ of $\Pi$ with parameter $k$, produces an **equivalent** instance $I'$ of $\Pi$ with parameter $k'$ such that $|I'| \leq f(k)$ and $k' \leq f(k)$ for a computable function $f$.
We refer to the function $f$ as the size of the kernel.

# Fixed-parameter tractability

## Definition 2

A parameterized problem $\Pi$ is fixed-parameter tractable (FPT) if there is an algorithm solving $\Pi$ in time $f(k) \cdot \text{poly}(n)$, where $n$ is the instance size, $k$ is the parameter, poly is a polynomial function, and $f$ is a computable function.

## Theorem 3

*Let $\Pi$ be a decidable parameterized problem.*
$\Pi$ *has a kernelization* $\Leftrightarrow$ $\Pi$ *is* FPT.

# Outline

# Outline

# Hamiltonian Cycle I

A Hamiltonian cycle of $G$ is a subgraph of $G$ that is a cycle on $|V(G)|$ vertices.

| vc-Hamiltonian Cycle | |
|---|---|
| Input: | A graph $G = (V, E)$. |
| Parameter: | $k = vc(G)$, the size of a smallest vertex cover of $G$. |
| Question: | Does $G$ have a Hamiltonian cycle? |

**Thought experiment**: Imagine a very large instance where the parameter is tiny. How can you simplify such an instance?

**Issue**: We do not actually know a vertex cover of size $k$.

# Hamiltonian Cycle III

- Obtain a vertex cover of size $\leq 2k$ by applying Vertex Cover-kernelizations to $(G, 0), (G, 1), \ldots$ until the first instance where no trivial No-instance is returned.
- If $C$ is a vertex cover of size $\leq 2k$, then $I = V \setminus C$ is an independent set of size $\geq |V| - 2k$.
- No two consecutive vertices in the Hamiltonian Cycle can be in $I$.
- A kernel with $\leq 4k$ vertices can now be obtained with the following simplification rule.

**(Too-large)**

Compute a vertex cover $C$ of size $\leq 2k$ in polynomial time.
If $2|C| < |V|$, then return No

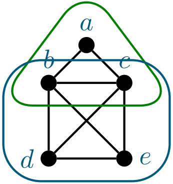
# Outline

# Edge Clique Cover

## Definition 4

An edge clique cover of a graph $G = (V, E)$ is a set of cliques in $G$ covering all its edges.
In other words, if $\mathcal{C} \subseteq 2^V$ is an edge clique cover then each $S \in \mathcal{C}$ is a clique in $G$ and for each $\{u, v\} \in E$ there exists an $S \in \mathcal{C}$ such that $u, v \in S$.

Example: $\{\{a, b, c\}, \{b, c, d, e\}\}$ is an edge clique cover for this graph.



# Edge Clique Cover

| | |
|---|---|
| **Edge Clique Cover** | |
| Input: | A graph $G = (V, E)$ and an integer $k$ |
| Parameter: | $k$ |
| Question: | Does $G$ have an edge clique cover of size at most $k$? |

The size of an edge clique cover $\mathcal{C}$ is the number of cliques contained in $\mathcal{C}$ and is denoted $|\mathcal{C}|$.

# Helpful properties

## Definition 4

A clique $S$ in a graph $G$ is a maximal clique if there is no other clique $S'$ in $G$ with $S \subset S'$.

## Lemma 5

*A graph $G$ has an edge clique cover $\mathcal{C}$ of size at most $k$ if and only if $G$ has an edge clique cover $\mathcal{C}'$ of size at most $k$ such that each $S \in \mathcal{C}'$ is a maximal clique.*

## Proof sketch.

$(\Rightarrow)$: Replace each clique $S \in \mathcal{C}$ by a maximal clique $S'$ with $S \subseteq S'$.
$(\Leftarrow)$: Trivial, since $\mathcal{C}'$ is an edge clique cover of size at most $k$. □

# Simplification rules for Edge Clique Cover

**Thought experiment**: Imagine a very large instance where the parameter is tiny. How can you simplify such an instance?

# Simplification rules for EDGE CLIQUE COVER II

The instance could have many degree-0 vertices.

**(Isolated)**

If there exists a vertex $v \in V$ with $d_G(v) = 0$, then set $G \leftarrow G - v$.

**Lemma 6**

*(Isolated) is sound.*

**Proof sketch.**

Since no edge is incident to $v$, a smallest edge clique cover for $G - v$ is a smallest edge clique cover for $G$, and vice-versa. □

# Simplification rules for Edge Clique Cover II

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**(Isolated-Edge)**

If $\exists uv \in E$ such that $d_G(u) = d_G(v) = 1$, then set $G \leftarrow G - \{u, v\}$ and $k \leftarrow k - 1$.

# Simplification rules for EDGE CLIQUE COVER III

## (Twins)

If $\exists u, v \in V$, $u \neq v$, such that $N_G[u] = N_G[v]$, then set $G \leftarrow G - v$.

## Lemma 7

*(Twins) is sound.*

# Simplification rules for Edge Clique Cover III

## (Twins)

If $\exists u, v \in V$, $u \neq v$, such that $N_G[u] = N_G[v]$, then set $G \leftarrow G - v$.

## Lemma 7

*(Twins) is sound.*

## Proof.

We need to show that $G$ has an edge clique cover of size at most $k$ if and only if $G - v$ has an edge clique cover of size at most $k$.
$(\Rightarrow)$: If $\mathcal{C}$ is an edge clique cover of $G$ of size at most $k$, then $\{S \setminus \{v\} : S \in \mathcal{C}\}$ is an edge clique cover of $G - v$ of size at most $k$.
$(\Leftarrow)$: Let $\mathcal{C}'$ be an edge clique cover of $G - v$ of size at most $k$. Partition $\mathcal{C}$ into $\mathcal{C}_u = \{S \in \mathcal{C} : u \in S\}$ and $\mathcal{C}_{\neg u} = \mathcal{C} \setminus \mathcal{C}_u$. Note that each set in $\mathcal{C}'_u = \{S \cup \{v\} : S \in \mathcal{C}_u\}$ is a clique since $N_G[u] = N_G[v]$ and that each edge incident to $v$ is contained in at least one of these cliques. Now, $\mathcal{C}'_u \cup \mathcal{C}_{\neg u}$ is an edge clique cover of $G$ of size at most $k$. □

# Simplification rules for Edge Clique Cover IV

**(Size-V)**

If the previous simplification rules do not apply and $|V| > 2^k$, then return No.

**Lemma 8**

*(Size-V) is sound.*

# Simplification rules for Edge Clique Cover IV

## (Size-V)

If the previous simplification rules do not apply and $|V| > 2^k$, then return No.

## Lemma 8

*(Size-V) is sound.*

## Proof.

For the sake of contradiction, assume neither (Isolated) nor (Twins) are applicable, $|V| > 2^k$, and $G$ has an edge clique cover $\mathcal{C}$ of size at most $k$. Since $2^{\mathcal{C}}$ (the set of all subsets of $\mathcal{C}$) has size at most $2^k$, and every vertex belongs to at least one clique in $\mathcal{C}$ by (Isolated), we have that there exists two vertices $u, v \in V$ such that $\{S \in \mathcal{C} : u \in S\} = \{S \in \mathcal{C} : v \in S\}$. But then, $N_G[u] = \bigcup_{S \in \mathcal{C} : u \in S} S = \bigcup_{S \in \mathcal{C} : v \in S} S = N_G[v]$, contradicting that (Twin) is not applicable. □

# Kernel for EDGE CLIQUE COVER

**Theorem 9**

EDGE CLIQUE COVER *has a kernel with* $O(2^k)$ *vertices and* $O(4^k)$ *edges.*

**Corollary 10**

EDGE CLIQUE COVER *is* FPT.

# Outline

# Frequent issues designing simplification rules

**Issue 1**: A kernelization needs to produce an instance of the same problem.

How could we turn the following lemma into a simplification rule?

## Lemma 11

*If there is an edge* $\{u, v\} \in E$ *such that* $S = N_G[u] \cap N_G[v]$ *is a clique, then there is a smallest edge clique cover* $\mathcal{C}$ *with* $S \in \mathcal{C}$.

## Proof.

By Lemma 5, we may assume the clique covering the edge $\{u, v\}$ is a maximal clique. But, $S$ is the unique maximal clique covering $\{u, v\}$. □

# Frequent issues designing simplification rules

**(Neighborhood-Clique)**

If there exists $\{u, v\} \in E$ such that $S = N_G[u] \cap N_G[v]$ is a clique, then ...???

Edges with both endpoints in $S \setminus \{u, v\}$ are covered by $S$ but might still be needed in other cliques.

# Frequent issues designing simplification rules

We could design a kernelization for a more general problem.

Generalized Edge Clique Cover

| | |
|---|---|
| Input: | A graph $G = (V, E)$, a set of edges $R \subseteq E$, and an integer $k$ |
| Parameter: | $k$ |
| Question: | Is there a set $\mathcal{C}$ of at most $k$ cliques in $G$ such that each $e \in R$ is contained in at least one of these cliques? |

**(Neighborhood-Clique)**

If there exists $\{u, v\} \in R$ such that $S = N_G[u] \cap N_G[v]$ is a clique, then set $G \leftarrow (V, E \setminus \{u, v\})$, $R \leftarrow R \setminus \{\{x, y\} : x, y \in S\}$, and $k \leftarrow k - 1$.

# Frequent issues designing simplification rules

**Issue 2**: A proposed simplification rule might not be sound.

Consider the following simplification rule for VERTEX COVER.

**(Optimistic-Degree-($\geq k$))**

If $\exists v \in V$ such that $d_G(v) \geq k$, then set $G \leftarrow G - v$ and $k \leftarrow k - 1$.

To show that a simplification rule is not sound, we exhibit a counter-example.

**Lemma 11**

*(Optimistic-Degree-($\geq k$)) is not sound for* VERTEX COVER.

# Frequent issues designing simplification rules

## (Optimistic-Degree-($\geq k$))

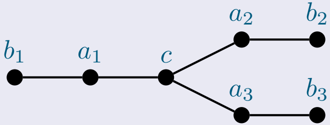
If $\exists v \in V$ such that $d_G(v) \geq k$, then set $G \leftarrow G - v$ and $k \leftarrow k - 1$.

## Lemma 11

*(Optimistic-Degree-($\geq k$)) is not sound for* VERTEX COVER*.*

## Proof.

Consider the instance consisting of the following graph and $k = 3$.




Since $M = \{\{a_i, b_i\} : 1 \leq i \leq 3\}$ is a matching, a vertex cover contains at least one endpoint of each edge in $M$. The rule would add $c$ to the vertex cover, leading to a vertex cover of size at least $4$. However, $\{a_i : 1 \leq i \leq 3\}$ is a vertex cover of size 3. □

# Frequent issues designing simplification rules

**Issue 3**: A problem might be FPT, but only an exponential kernel might be known / possible to achieve.

# Outline

# Polynomial vs. exponential kernels

- For some FPT problems, only exponential kernels are known.
- Could it be that all FPT problems have polynomial kernels?
- We will see that polynomial kernels for some fixed-parameter tractable parameterized problems would contradict complexity-theoretic assumptions.

# Intuition by example

| | |
|---|---|
| **Long Path** | |
| Input: | A graph $G = (V, E)$, and an integer $k \leq \|V\|$. |
| Parameter: | $k$ |
| Question: | Does $G$ have a path of length at least $k$ (as a subgraph)? |

Long Path is NP-complete but FPT.

# Intuition by example

- Assume Long Path has a $k^c$ kernel, where $c = O(1)$.
- Set $q = k^c + 1$ and consider $q$ instances with the same parameter $k$:

$$(G_1, k), (G_2, k), \ldots, (G_q, k).$$

- Let $G = G_1 \oplus G_2 \oplus \cdots \oplus G_q$ be the disjoint union of all these graphs.
- Note that $(G, k)$ is a Yes-instance if and only if at least one of $(G_i, k), 1 \leq i \leq q$, is a Yes-instance.
- Kernelizing $(G, k)$ gives an instance of size $k^c$, i.e., on average less than one bit per original instance.
- "The kernelization must have solved at least one of the original NP-hard instances in polynomial time".
- Note that this is not a rigorous argument, and we will make this more formal now.

# Outline

# Distillation

## Definition 11

Let $\Pi_1, \Pi_2$ be two problems. An OR-distillation (resp., AND-distillation) from $\Pi_1$ into $\Pi_2$ is a polynomial time algorithm $D$ whose input is a sequence $I_1, \ldots, I_q$ of instances for $\Pi_1$ and whose output is an instance $I'$ for $\Pi_2$ such that

- $|I'| \leq \text{poly}(\max_{1 \leq i \leq q} |I_i|)$, and
- $I'$ is a $\textsc{Yes}$-instance for $\Pi_2$ if and only if for at least one (resp., for each) $i \in \{1, \ldots, q\}$ we have that $I_i$ is a $\textsc{Yes}$-instance for $\Pi_1$.

# NP-complete problems don't have distillations

**Theorem 12 ([Fortnow, Santhanam, 2008])**

*If any NP-complete problem has an OR-distillation, then coNP $\subseteq$ NP/poly.* [1]

**Note**: coNP $\subseteq$ NP/poly is not believed to be true and it would imply that the polynomial hierarchy collapses to the third level: PH $\subseteq \Sigma_3^p$.

**Theorem 13 ([Drucker, 2012])**

*If any NP-complete problem has an AND-distillation, then coNP $\subseteq$ NP/poly.*

[1] NP/poly is the class of all decision problems for which there exists a polynomial-time nondeterministic Turing Machine $M$ with the following property: for every $n \geq 0$, there is an advice string $A$ of length $\text{poly}(n)$ such that, for every input $I$ of length $n$, the machine $M$ correctly decides the problem with input $I$, given $I$ and $A$.

# Composition algorithms

## Definition 14

Let $\Pi$ be a parameterized problem. An OR-composition (resp., AND-composition) of $\Pi$ is a polynomial time algorithm $A$ that receives as input a finite sequence $I_1, \dots, I_q$ of $\Pi$ with parameters $k_1 = \dots = k_q = k$ and outputs an instance $I'$ for $\Pi$ with parameter $k'$ such that

- $k' \leq \text{poly}(k)$, and
- $I'$ is a YES-instance for $\Pi$ if and only if for at least one (resp., for each) $i \in \{1, \dots, q\}$, $I_i$ is a YES-instance for $\Pi$.

# Tool for showing kernel lower bounds

**Theorem 15 (Composition Theorem)**

*Let $\Pi$ be an NP-complete parameterized problem such that for each instance $I$ of $\Pi$ with parameter $k$, the value of the parameter $k$ can be computed in polynomial time and $k \leq |I|$. If $\Pi$ has an OR-composition or an AND-composition, then $\Pi$ has no polynomial kernel, unless coNP $\subseteq$ NP/poly.*

# Tool for showing kernel lower bounds

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## Proof sketch.

Suppose $\Pi$ has an OR/AND-composition and a polynomial kernel. Then, one can obtain an OR/AND-distillation from $\Pi$ into OR($\Pi$)/AND($\Pi$).

# Tool for showing kernel lower bounds

## Theorem 15 (Composition Theorem)

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## Proof sketch.

Suppose $\Pi$ has an OR/AND-composition and a polynomial kernel. Then, one can obtain an OR/AND-distillation from $\Pi$ into OR($\Pi$)/AND($\Pi$).

| | |
|---|---|
| $I_1 \quad I_2 \quad \dots \quad I_q$ | $q$ instances of size $\le n = \max_{1 \le i \le q} \lvert I_i \rvert$ |
| $\{I_i : k_i = 0\} \dots \{I_i : k_i = n\}$ | group by parameter |
| $I'_0 \quad I'_1 \quad \dots \quad I'_n$ | After OR-composition: $n+1$ instances with $k'_i \le$ poly$(n)$ |
| $I''_0 \quad I''_1 \quad \dots \quad I''_n$ | After kernelization: $n+1$ instances of size poly$(n)$ each |
| | This is an instance of OR($\Pi$) of size poly$(n)$. |

□

# LONG PATH has no polynomial kernel I

**Theorem 16**

LONG PATH *has no polynomial kernel unless* NP $\subseteq$ coNP/poly.

**Proof.**

Clearly, $k$ can be computed in polynomial time and $k \leq |V|$.
We give an OR-composition for LONG PATH, which will prove the theorem by the previous lemma.
It receives as input a sequence of instances for LONG PATH: $(G_1, k), \ldots, (G_q, k)$, and it produces the instance $(G_1 \oplus \cdots \oplus G_q, k)$, which is a YES-instance if and only if at least one of $(G_1, k), \ldots, (G_q, k)$ is a YES-instance. □

# var-$\mathrm{SAT}$ has no poly kernel I

var-$\mathrm{SAT}$

| | |
|---|---|
| Input: | A propositional formula $F$ in conjunctive normal form (CNF) |
| Parameter: | $n = \lvert \mathsf{var}(F) \rvert$, the number of variables in $F$ |
| Question: | Is there an assignment to $\mathsf{var}(F)$ satisfying all clauses of $F$? |

**Example**:

$$(x_1 \vee x_2) \wedge (\neg x_2 \vee x_3 \vee \neg x_4) \wedge (x_1 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$$

or

$$\{\{x_1, x_2\}, \{\neg x_2, x_3, \neg x_4\}, \{x_1, x_4\}, \{\neg x_1, \neg x_3, \neg x_4\}\}$$

# var-$\mathrm{SAT}$ has no poly kernel II

**Theorem 17**

*var-$\mathrm{SAT}$ has no polynomial kernel unless* NP $\subseteq$ coNP/poly.

**Proof.**

Clearly, var$(F)$ can be computed in polynomial time and $n = |\text{var}(F)| \leq |F|$. We give an OR-composition for var-$\mathrm{SAT}$, which will prove the theorem by the previous lemma.

- Let $F_1, \ldots, F_q$ be CNF formulas, $|F_i| \leq m$, $|\text{var}(F_i)| = n$.
- We can decide whether one of the formulas is satisfiable in time poly$(mt2^n)$. Hence, if $q > 2^n$, the check is polynomial. If some formula is satisfiable, we output this formula, otherwise we output $F_1$.

# var-SAT has no poly kernel III

## Proof (continued).

- It remains the case $q \leq 2^n$. We assume $\text{var}(F_1) = \cdots = \text{var}(F_q)$, otherwise we change the names of variables.
- Let $s = \lceil \log_2 q \rceil$. Since $q \leq 2^n$, we have that $s \leq n$.
- We take a set $Y = \{y_1, \ldots, y_s\}$ of new variables. Let $C_1, \ldots, C_{2^s}$ be the sequence of all $2^s$ possible clauses containing exactly $s$ literals over the variables in $Y$.
- For $1 \leq i \leq q$ we let $F_i' = \{C \cup C_i : C \in F_i\}$.
- We define $F = \bigcup_{i=1}^{q} F_i' \cup \{C_i : q+1 \leq i \leq 2^s\}$.
- Claim: $F$ is satisfiable if and only if $F_i$ is satisfiable for some $1 \leq i \leq q$.
- Hence we have an OR-composition.

□

# Outline

# Another tool for showing kernel lower bounds I

## Definition 18

Let $\Pi_1, \Pi_2$ be parameterized problems. A polynomial parameter transformation from $\Pi_1$ to $\Pi_2$ is a polynomial time algorithm, which, for any instance $I_1$ of $\Pi_1$ with parameter $k_1$, produces an **equivalent** instance $I_2$ of $\Pi_2$ with parameter $k_2$ such that $k_2 \leq \text{poly}(k_1)$.

# Another tool for showing kernel lower bounds II

**Theorem 19**

*Let $\Pi_1, \Pi_2$ be parameterized problems such that $\Pi_1$ is NP-complete, $\Pi_2$ is in NP, and there is a polynomial parameter transformation from $\Pi_1$ to $\Pi_2$. If $\Pi_2$ has a polynomial kernel, then $\Pi_1$ has a polynomial kernel.*

**Remark**: If we know that an NP-complete parameterized problem $\Pi_1$ has no polynomial kernel (unless NP $\subseteq$ coNP/poly), we can use the theorem to show that some other NP-complete parameterized problem $\Pi_2$ has no polynomial kernel (unless NP $\subseteq$ coNP/poly) by giving a polynomial parameter transformation from $\Pi_1$ to $\Pi_2$.

# Another tool for showing kernel lower bounds III

## Proof.

- We show that under the assumptions of the theorem $\Pi_1$ has a polynomial kernel.
- Let $I_1$ be an instance of $\Pi_1$ with parameter $k_1$.
- We obtain in polynomial time an equivalent instance $I_2$ of $\Pi_2$ with parameter $k_2 \leq \text{poly}(k_1)$.
- We apply $\Pi_2$'s kernelization and obtain $I_2'$ of size $\leq \text{poly}(k_1)$.
- Since $\Pi_2$ is in NP and $\Pi_1$ is NP-complete, there exists a polynomial time reduction that maps $I_2'$ to an equivalent instance $I_1'$ of $\Pi_1$.
- The size of $I_1'$ is polynomial in $k_1$.

□

**Definition 20**

A CNF formula $F$ is a 2CNF formula if each clause of $F$ has at most 2 literals.

**Note**: SAT is polynomial time solvable when the input is restricted to be a 2CNF formula.

**Definition 21**

A 2CNF-backdoor of a CNF formula $F$ is a set of variables $B \subseteq \text{var}(F)$ such that for each assignment $\alpha : B \rightarrow \{0, 1\}$, the formula $F[\alpha]$ is a 2CNF formula.
Here, $F[\alpha]$ is obtained by removing all clauses containing a literal set to $1$ by $\alpha$, and removing the literals set to $0$ from all remaining clauses.

# 2CNF-Backdoor Evaluation II

| 2CNF-Backdoor Evaluation | |
|---|---|
| Input: | A CNF formula $F$ and a 2CNF-backdoor $B$ of $F$ |
| Parameter: | $k = \|B\|$ |
| Question: | Is $F$ satisfiable? |

**Note**: the problem is FPT by trying all assignments to $B$ and evaluating the resulting formulas.

# 2CNF-Backdoor Evaluation III

**Theorem 22**

2CNF-Backdoor Evaluation *has no polynomial kernel unless* NP $\subseteq$ coNP/poly.

**Proof.**

We give a polynomial parameter transformation from var-SAT to 2CNF-Backdoor Evaluation.
Let $F$ be an instance for var-SAT.
Then, $(F, B = \text{var}(F))$ is an equivalent instance for 2CNF-Backdoor Evaluation with $|B| \leq |\text{var}(F)|$. □

# Outline

# Further Reading

- Chapter 15, *Lower bounds for kernelization* in
  Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, MichałPilipczuk, and Saket Saurabh. Parameterized Algorithms. Springer, 2015.
- Chapter 30 (30.1–30.4), *Kernelization Lower Bounds* in
  Rodney G. Downey and Michael R. Fellows. Fundamentals of Parameterized Complexity. Springer, 2013.
- Neeldhara Misra, Venkatesh Raman, and Saket Saurabh. *Lower bounds on kernelization*. Discrete Optimization 8(1): 110-128 (2011).