COMP2111 Week 11
Term 1, 2019
Q & A

Administrivia

- Assignment 3 due today, 23:59. Solution next week.
- Assignment 2 marks released soon
- Problem set solutions released later this week

Post-course consultations (Me):

- Room 204, K17
- Today 12:00-1:00
- Monday, May 6: 11:00-4:00

Topics covered today

1. Hoare Logic
2. CFGs / Myhill-Nerode
3. Transition systems
4. Natural Deduction (Predicate Logic)
5. DFAs/NFAs/Regular languages

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Hoare Logic

Problem 8

$$
\begin{array}{ll}
 & \{n \geq 0\} \\
 & \{A\} \\
q := 0; & \{B\} \\
r := n; & \{C\} \\
\textbf{while } r \geq 3 \textbf{ do} & \{D\} \\
 & \{E\} \\
\quad q := q + 1; & \{F\} \\
\quad r := r - 3 & \{G\} \\
\textbf{od} & \{H\} \\
 & \{\mathsf{Post}\}
\end{array}
$$

(a) Find Post, relating q, r, and n.

(b) Complete the annotated proof by finding suitable assertions for A–H; include proof obligations.

Hoare Logic solution

Problem 8 solution

Post	$q = n \text{ div } 3 \wedge r = n \bmod 3$
C	$(n = 3q + r) \wedge (r \geq 0)$
G	$(n = 3q + r) \wedge (r \geq 0)$
D	$(n = 3q + r) \wedge (r \geq 0) \wedge (r \geq 3)$
H	$(n = 3q + r) \wedge (r \geq 0) \wedge (r < 3)$
F	$(n = 3q + (r - 3)) \wedge ((r - 3) \geq 0)$
E	$(n = 3(q + 1) + (r - 3)) \wedge ((r - 3 \geq 0)$
B	$(n = 3q + n) \wedge (n \geq 0)$
A	$(n = 3.0 + n) \wedge (n \geq 0)$

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CFGs / Myhill-Nerode

Problem 11

Consider the language
$L = \{w \in \{0,1\}^* \ : \ w \text{ has more 0's than 1's}\}$.

(a) Give a context-free grammar that generates L

(b) Use the Myhill-Nerode theorem to show that L is not regular.

CFGs / Myhill-Nerode solution

Problem 11 solution

(a) L is generated by the grammar $(\{W, Z, S\}, \{0, 1\}, \mathcal{R}, S)$ where $\mathcal{R}$ is the ruleset:

$$\begin{array}{rcl} S & \rightarrow & WZS \mid WZW \\ W & \rightarrow & WW \mid 0W1 \mid 1W0 \mid \epsilon \\ Z & \rightarrow & 0Z \mid 0 \end{array}$$

(b) Let $w_i = 0^i$ for $i \geq 1$. We will show that if $i \neq j$ then $w_i \not\equiv_L w_j$. Assume, without loss of generality that $i < j$. Then for $z = 1^i$ we have $w_i z \notin L$ but $w_j z \in L$ so $w_i \not\equiv_L w_j$. Therefore L has infinite index, so by the Myhill-Nerode theorem it is not regular.

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Problem 9

A knight can make one of eight possible moves: from (x, y) it can move to any of:

(i) $(x+2, y+1)$

(ii) $(x+2, y-1)$

(iii) $(x+1, y+2)$

(iv) $(x+1, y-2)$

(v) $(x-1, y-2)$

(vi) $(x-1, y+2)$

(vii) $(x-2, y-1)$

(viii) $(x-2, y+1)$

Transition systems

Problem 9

(a) Model this process as a state transition system, defining the states and transition relation.

Transition systems solution

Problem 9(a) solution

States: $\mathbb{Z} \times \mathbb{Z}$

Transition: A transition from (x, y) to each of:

- **(i)** $(x + 2, y + 1)$
- **(ii)** $(x + 2, y - 1)$
- **(iii)** $(x + 1, y + 2)$
- **(iv)** $(x + 1, y - 2)$
- **(v)** $(x - 1, y - 2)$
- **(vi)** $(x - 1, y + 2)$
- **(vii)** $(x - 2, y - 1)$
- **(viii)** $(x - 2, y + 1)$

Transition systems

Problem 9

(b) Prove that from $(0, 0)$ the knight is able to reach any location (x, y) in a finite number of moves.

(c) Based on your previous answer, give an upper bound for the number of moves it takes the knight to reach (x, y) from $(0, 0)$

Transition systems solution

Problem 9(b)&(c) solutions

$$
\begin{array}{ccccccc}
(x,y) & \xrightarrow{(iii)} & (x+1,y+2) & \xrightarrow{(vii)} & (x-1,y+1) & \xrightarrow{(ii)} & (x+1,y) \\
(x,y) & \xrightarrow{(vi)} & (x-1,y+2) & \xrightarrow{(ii)} & (x+1,y+1) & \xrightarrow{(vii)} & (x-1,y) \\
(x,y) & \xrightarrow{(i)} & (x+2,y+1) & \xrightarrow{(v)} & (x+1,y-1) & \xrightarrow{(vi)} & (x,y+1) \\
(x,y) & \xrightarrow{(ii)} & (x+2,y-1) & \xrightarrow{(vi)} & (x+1,y+1) & \xrightarrow{(v)} & (x,y-1)
\end{array}
$$

Given these compound moves, it is straightforward to see how the knight can reach any square from (x,y) with a suitable combination of N, S, E, W moves.

Move from $(0,0)$ to (x,y) takes at most $3(x+y)$ moves

Transition systems

Problem 9

(d) The *kih* is restricted to moves (i), (iii), (v) and (vii). Find an invariant for the set of states reachable from $(0, 0)$ by a kih; and show that $(1, 0)$ is not reachable from $(0, 0)$.

(e) The *ngt* is restricted to moves (ii), (iv), (vi) and (viii). Show that a ngt cannot reach $(1, 0)$ from $(0, 0)$.

Transition systems solution

Problem 9(d)&(e) solutions

(d)

- Consider: "$x + y \pmod 3 = 0$".
- It holds at $(0, 0)$
- If $(x, y) \xrightarrow{kih} (x', y')$ then $x' + y' = x + y \pm 3$.
- So it is a preserved invariant of the states reachable from $(0, 0)$.
- At $(1, 0)$: $1 + 0 \neq 0 \pmod 3$, it follows that $(1, 0)$ is not reachable from $(0, 0)$ by a kih.

(e)

- Consider: "$x - y \pmod 3 = 0$".
- It holds at $(0, 0)$
- If $(x, y) \xrightarrow{ngt} (x', y')$ then $x' - y' = x - y \pm 3$.
- So it is a preserved invariant of the states reachable from $(0, 0)$.
- At $(1, 0)$: $1 - 0 \neq 0 \pmod 3$, it follows that $(1, 0)$ is not reachable from $(0, 0)$ by a ngt.

Transition systems

Problem 9

(f) The *kni* is restricted to moves (i), (ii), (iii) and (iv). Show that a kni cannot reach $(1, 0)$ from $(0, 0)$.

Transition systems solution

Problem 9(d)&(e) solutions

Keep track of moves made:

States: $\mathbb{N} \times \mathbb{Z} \times \mathbb{Z}$

Transition: A transition from (n, x, y) to each of:

- **(i)** $(n+1, x+2, y+1)$
- **(ii)** $(n+1, x+2, y-1)$
- **(iii)** $(n+1, x+1, y+2)$
- **(iv)** $(n+1, x+1, y-2)$

Invariant: At (n, x, y): $x \geq n$

Check all reachable positions in 0 or 1 moves.

Transition systems solution

Problem 9(d)&(e) solutions

Keep track of moves made:

States: $\mathbb{N} \times \mathbb{Z} \times \mathbb{Z}$

Transition: A transition from (n, x, y) to each of:

- **(i)** $(n+1, x+2, y+1)$
- **(ii)** $(n+1, x+2, y-1)$
- **(iii)** $(n+1, x+1, y+2)$
- **(iv)** $(n+1, x+1, y-2)$

Invariant: At (n, x, y): $x \geq n$

Check all reachable positions in 0 or 1 moves.

Transition systems solution

Problem 9(d)&(e) solutions

Keep track of moves made:

States: $\mathbb{N} \times \mathbb{Z} \times \mathbb{Z}$

Transition: A transition from (n, x, y) to each of:

- **(i)** $(n+1, x+2, y+1)$
- **(ii)** $(n+1, x+2, y-1)$
- **(iii)** $(n+1, x+1, y+2)$
- **(iv)** $(n+1, x+1, y-2)$

Invariant: At (n, x, y): $x \geq n$

Check all reachable positions in 0 or 1 moves.

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Recall:

Problems 2 and 5:

2(a) Suppose $X \cap Y^c = \emptyset$ and $Y \cap Z = \emptyset$. Show:

$$X \cap Z = \emptyset$$

2(b) Suppose $X \cap Y = \emptyset$. Show that for all Z:

$$(Z \cap X^c) \cap (Z \cap Y) = (Z \cap Y)$$

5(a) If all apples are fruit and no fruit are vegetables, then no apples are vegetables.

5(b) If no bananas are apples and pink-lady is an apple, then pink-lady is not a banana.

Natural Deduction

Problem 7:

Show

(a) $\vdash \forall x(A(x) \to F(x)) \wedge \neg\exists x(F(x) \wedge V(x)) \to \neg\exists x(A(x) \wedge V(x))$

(b) $\vdash \neg\exists x(B(x) \wedge A(x)) \wedge A(p) \to \neg B(p)$

Natural Deduction solutions

Problem 7(a) solution:

Line	Premises	Formula	Rule	Ref.
1		$\varphi_1 \wedge \varphi_2$	Premise	
2	1	$\varphi_1 : \forall x(A(x) \to F(x))$	$\wedge$-E1	1
3	1	$\varphi_2 : \neg\exists x(F(x) \wedge V(x))$	$\wedge$-E2	1
4		$\exists x(A(x) \wedge V(x))$	Premise	
5		$A(c) \wedge V(c)$	Premise	
6	5	$A(c)$	$\wedge$-E1	5
7	1	$A(c) \to F(c)$	$\forall$-E	2
8	1,5	$F(c)$	$\to$-E	6,7
9	5	$V(c)$	$\wedge$-E2	5
10	1,5	$F(c) \wedge V(c)$	$\wedge$-I	8,9
11	1,5	$\exists x(F(x) \wedge V(x))$	$\exists$-I	10
12	1,5	$\bot$	$\neg$-E	3,11
13	1,4	$\bot$	$\exists$-E	4,12
14	1	$\neg\exists x(A(x) \wedge V(x))$	$\neg$-I	12
15		$\varphi_1 \wedge \varphi_2 \to \neg\exists x(A(x) \wedge V(x))$	$\to$-I	13

Natural Deduction solutions

Problem 7(b) solution:

1. $\neg\exists x(B(x) \wedge A(x)) \wedge A(p)$	
2. $\neg\exists x(B(x) \wedge A(x))$	$\wedge$-E1: 1
3. $A(p)$	$\wedge$-E2: 1
4. $B(p)$	
5. $B(p) \wedge A(p)$	$\wedge$-I: 3,4
6. $\exists x(B(x) \wedge A(x))$	$\exists$-I: 5
7. $\bot$	$\neg$-E: 2,6
8. $\neg B(p)$	
9. $\neg\exists x(B(x) \wedge A(x)) \wedge A(p) \rightarrow \neg B(p)$	$\rightarrow$-I: 1-8

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DFAs/NFAs/Regular languages

Problem 10:

Give DFAs that accepts the following languages over $\Sigma = \{a, b\}$:

(a) $L_1 = \{\lambda\} \cup \{w \ : \ w \text{ starts and ends with the same symbol}\}$

(b) $L_2 = \{a, aa, bb, aba, bab, bba\}$

(c) $L_3 = \{w \ :$
w has an even number of a's and an odd number of b's$\}$

(d) $L_1 \cap L_3$

Conclusion

Any other questions?