COMP2111 Week 11
Term 1, 2019
Q & A
Assignment 3 due today, 23:59. Solution next week.
Assignment 2 marks released soon
Problem set solutions released later this week

Post-course consultations (Me):
Room 204, K17
Today 12:00-1:00
Monday, May 6: 11:00-4:00
Topics covered today

1. Hoare Logic
2. CFGs / Myhill-Nerode
3. Transition systems
4. Natural Deduction (Predicate Logic)
5. DFAs/NFAs/Regular languages
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Problem 8

\[
\begin{align*}
\{ n \geq 0 \} & \quad \{ A \} \\
q & := 0; & \quad \{ B \} \\
r & := n; & \quad \{ C \} \\
{\textbf{while} \ r \geq 3 \ {\textbf{do}}} & \quad \{ D \} \\
& \quad \{ E \} \\
q & := q + 1; & \quad \{ F \} \\
r & := r - 3 & \quad \{ G \} \\
{\textbf{od}} & \quad \{ H \} \\
\end{align*}
\]

(a) Find Post, relating \( q \), \( r \), and \( n \).

(b) Complete the annotated proof by finding suitable assertions for \( A-H \); include proof obligations.
Problem 8 solution

Post \[ q = n \text{ div } 3 \land r = n \text{ mod } 3 \]

\[ C \quad (n = 3q + r) \land (r \geq 0) \]

\[ G \quad (n = 3q + r) \land (r \geq 0) \]

\[ D \quad (n = 3q + r) \land (r \geq 0) \land (r \geq 3) \]

\[ H \quad (n = 3q + r) \land (r \geq 0) \land (r < 3) \]

\[ F \quad (n = 3q + (r - 3)) \land ((r - 3) \geq 0) \]

\[ E \quad (n = 3(q + 1) + (r - 3)) \land ((r - 3) \geq 0) \]

\[ B \quad (n = 3q + n) \land (n \geq 0) \]

\[ A \quad (n = 3.0 + n) \land (n \geq 0) \]
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Problem 11
Consider the language
\[ L = \{ w \in \{0, 1\}^* : w \text{ has more 0's than 1's} \}. \]
(a) Give a context-free grammar that generates \( L \)
(b) Use the Myhill-Nerode theorem to show that \( L \) is not regular.
Problem 11 solution

(a) $L$ is generated by the grammar $(\{W, Z, S\}, \{0, 1\}, R, S)$ where $R$ is the ruleset:

$$
S \rightarrow WZS \mid WZW \\
W \rightarrow WW \mid 0W1 \mid 1W0 \mid \epsilon \\
Z \rightarrow 0Z \mid 0
$$

(b) Let $w_i = 0^i$ for $i \geq 1$. We will show that if $i \neq j$ then $w_i \not\equiv_L w_j$. Assume, without loss of generality that $i < j$. Then for $z = 1^i$ we have $w_i z \not\in L$ but $w_j z \in L$ so $w_i \not\equiv_L w_j$. Therefore $L$ has infinite index, so by the Myhill-Nerode theorem it is not regular.
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Problem 9

A knight can make one of eight possible moves: from \((x, y)\) it can move to any of:

(i) \((x + 2, y + 1)\)
(ii) \((x + 2, y - 1)\)
(iii) \((x + 1, y + 2)\)
(iv) \((x + 1, y - 2)\)
(v) \((x - 1, y - 2)\)
(vi) \((x - 1, y + 2)\)
(vii) \((x - 2, y - 1)\)
(viii) \((x - 2, y + 1)\)
Problem 9
(a) Model this process as a state transition system, defining the states and transition relation.
Transition systems solution

Problem 9(a) solution

States: \( \mathbb{Z} \times \mathbb{Z} \)

Transition: A transition from \((x, y)\) to each of:

(i) \((x + 2, y + 1)\)
(ii) \((x + 2, y - 1)\)
(iii) \((x + 1, y + 2)\)
(iv) \((x + 1, y - 2)\)
(v) \((x - 1, y - 2)\)
(vi) \((x - 1, y + 2)\)
(vii) \((x - 2, y - 1)\)
(viii) \((x - 2, y + 1)\)
Problem 9

(b) Prove that from (0, 0) the knight is able to reach any location (x, y) in a finite number of moves.

(c) Based on your previous answer, give an upper bound for the number of moves it takes the knight to reach (x, y) from (0, 0).
## Transition systems solution

### Problem 9(b)&(c) solutions

<table>
<thead>
<tr>
<th>Move</th>
<th>Transition</th>
<th>Source</th>
<th>Destination</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x, y)</td>
<td>(iii)</td>
<td>(x + 1, y + 2)</td>
<td>(x + 1, y)</td>
</tr>
<tr>
<td>(x, y)</td>
<td>(vi)</td>
<td>(x − 1, y + 2)</td>
<td>(x − 1, y)</td>
</tr>
<tr>
<td>(x, y)</td>
<td>(i)</td>
<td>(x + 2, y + 1)</td>
<td>(x, y + 1)</td>
</tr>
<tr>
<td>(x, y)</td>
<td>(ii)</td>
<td>(x + 2, y − 1)</td>
<td>(x, y − 1)</td>
</tr>
<tr>
<td>(x, y)</td>
<td>(vii)</td>
<td>(x − 1, y + 1)</td>
<td>(x + 1, y + 1)</td>
</tr>
<tr>
<td>(x, y)</td>
<td>(ii)</td>
<td>(x + 1, y + 1)</td>
<td>(x + 1, y)</td>
</tr>
<tr>
<td>(x, y)</td>
<td>(vii)</td>
<td>(x − 1, y)</td>
<td>(x + 1, y + 1)</td>
</tr>
<tr>
<td>(x, y)</td>
<td>(vi)</td>
<td>(x, y + 1)</td>
<td>(x, y − 1)</td>
</tr>
</tbody>
</table>

Given these compound moves, it is straightforward to see how the knight can reach any square from (x, y) with a suitable combination of \(N, S, E, W\) moves.

Move from (0, 0) to (x, y) takes at most \(3(x + y)\) moves.
Problem 9

(d) The $kih$ is restricted to moves (i), (iii), (v) and (vii). Find an invariant for the set of states reachable from $(0, 0)$ by a $kih$; and show that $(1, 0)$ is not reachable from $(0, 0)$.

(e) The $ngt$ is restricted to moves (ii), (iv), (vi) and (viii). Show that a $ngt$ cannot reach $(1, 0)$ from $(0, 0)$. 
Problem 9(d)&(e) solutions

(d) • Consider: “$x + y \pmod{3} = 0$”.
   • It holds at $(0, 0)$
     - If $(x, y) \xrightarrow{k_{ih}} (x', y')$ then $x' + y' = x + y \pm 3$.
     - So it is a preserved invariant of the states reachable from $(0, 0)$.
   • At $(1, 0)$: $1 + 0 \neq 0 \pmod{3}$, it follows that $(1, 0)$ is not reachable from $(0, 0)$ by a $k_{ih}$.

(e) • Consider: “$x - y \pmod{3} = 0$”.
   • It holds at $(0, 0)$
     - If $(x, y) \xrightarrow{n_{gt}} (x', y')$ then $x' - y' = x - y \pm 3$.
     - So it is a preserved invariant of the states reachable from $(0, 0)$.
   • At $(1, 0)$: $1 - 0 \neq 0 \pmod{3}$, it follows that $(1, 0)$ is not reachable from $(0, 0)$ by a $n_{gt}$.
Problem 9

(f) The *kni* is restricted to moves (i), (ii), (iii) and (iv). Show that a kni cannot reach \((1, 0)\) from \((0, 0)\).
Problem 9(d)&(e) solutions

Keep track of moves made:

**States:** \( \mathbb{N} \times \mathbb{Z} \times \mathbb{Z} \)

**Transition:** A transition from \((n, x, y)\) to each of:

(i) \((n + 1, x + 2, y + 1)\)
(ii) \((n + 1, x + 2, y - 1)\)
(iii) \((n + 1, x + 1, y + 2)\)
(iv) \((n + 1, x + 1, y - 2)\)

**Invariant:** At \((n, x, y)\): \(x \geq n\)

Check all reachable positions in 0 or 1 moves.
Problem 9(d) & (e) solutions

Keep track of moves made:

**States:** \( \mathbb{N} \times \mathbb{Z} \times \mathbb{Z} \)

**Transition:** A transition from \((n, x, y)\) to each of:

(i) \((n + 1, x + 2, y + 1)\)
(ii) \((n + 1, x + 2, y - 1)\)
(iii) \((n + 1, x + 1, y + 2)\)
(iv) \((n + 1, x + 1, y - 2)\)

**Invariant:** At \((n, x, y)\): \( x \geq n \)

Check all reachable positions in 0 or 1 moves.
Problem 9(d) & (e) solutions

Keep track of moves made:

**States:** \( \mathbb{N} \times \mathbb{Z} \times \mathbb{Z} \)

**Transition:** A transition from \((n, x, y)\) to each of:

(i) \((n + 1, x + 2, y + 1)\)

(ii) \((n + 1, x + 2, y - 1)\)

(iii) \((n + 1, x + 1, y + 2)\)

(iv) \((n + 1, x + 1, y - 2)\)

**Invariant:** At \((n, x, y)\): \(x \geq n\)

Check all reachable positions in 0 or 1 moves.
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### Recall:

#### Problems 2 and 5:

2(a) Suppose $X \cap Y^c = \emptyset$ and $Y \cap Z = \emptyset$. Show:

$$X \cap Z = \emptyset$$

2(b) Suppose $X \cap Y = \emptyset$. Show that for all $Z$:

$$(Z \cap X^c) \cap (Z \cap Y) = (Z \cap Y)$$

5(a) If all apples are fruit and no fruit are vegetables, then no apples are vegetables.

5(b) If no bananas are apples and pink-lady is an apple, then pink-lady is not a banana.
Problem 7:

Show

(a) \( \vdash \forall x (A(x) \rightarrow F(x)) \land \neg \exists x (F(x) \land V(x)) \rightarrow \neg \exists x (A(x) \land V(x)) \)

(b) \( \vdash \neg \exists x (B(x) \land A(x)) \land A(p) \rightarrow \neg B(p) \)
### Natural Deduction solutions

#### Problem 7(a) solution:

<table>
<thead>
<tr>
<th>Line</th>
<th>Premises</th>
<th>Formula</th>
<th>Rule</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>( \varphi_1 \land \varphi_2 )</td>
<td>Premise</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>( \varphi_1 : \forall x (A(x) \rightarrow F(x)) )</td>
<td>( \land)-E1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>( \varphi_2 : \neg \exists x (F(x) \land V(x)) )</td>
<td>( \land)-E2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>( \exists x (A(x) \land V(x)) )</td>
<td>Premise</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>( A(c) \land V(c) )</td>
<td>Premise</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>( A(c) )</td>
<td>( \land)-E1</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>( A(c) \rightarrow F(c) )</td>
<td>( \forall)-E</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>1,5</td>
<td>( F(c) )</td>
<td>( \rightarrow)-E</td>
<td>6,7</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>( V(c) )</td>
<td>( \land)-E2</td>
<td>5</td>
</tr>
<tr>
<td>10</td>
<td>1,5</td>
<td>( F(c) \land V(c) )</td>
<td>( \land)-I</td>
<td>8,9</td>
</tr>
<tr>
<td>11</td>
<td>1,5</td>
<td>( \exists x (F(x) \land V(x)) )</td>
<td>( \exists)-I</td>
<td>10</td>
</tr>
<tr>
<td>12</td>
<td>1,5</td>
<td>( \bot )</td>
<td>( \neg)-E</td>
<td>3,11</td>
</tr>
<tr>
<td>13</td>
<td>1,4</td>
<td>( \bot )</td>
<td>( \exists)-E</td>
<td>4,12</td>
</tr>
<tr>
<td>14</td>
<td>1</td>
<td>( \neg \exists x (A(x) \land V(x)) )</td>
<td>( \neg)-I</td>
<td>12</td>
</tr>
<tr>
<td>15</td>
<td></td>
<td>( \varphi_1 \land \varphi_2 \rightarrow \neg \exists x (A(x) \land V(x)) )</td>
<td>( \rightarrow)-I</td>
<td>13</td>
</tr>
</tbody>
</table>
Problem 7(b) solution:

1. \( \neg \exists x (B(x) \land A(x)) \land A(p) \)
2. \( \neg \exists x (B(x) \land A(x)) \)
3. \( A(p) \)
4. \( B(p) \)
5. \( B(p) \land A(p) \)
6. \( \exists x (B(x) \land A(x)) \)
7. \( \bot \)
8. \( \neg B(p) \)
9. \( \neg \exists x (B(x) \land A(x)) \land A(p) \rightarrow \neg B(p) \)
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Problem 10:

Give DFAs that accepts the following languages over $\Sigma = \{a, b\}$:

(a) $L_1 = \{\lambda\} \cup \{w : w \text{ starts and ends with the same symbol}\}$

(b) $L_2 = \{a, aa, bb, aba, bab, bba\}$

(c) $L_3 = \{w : w \text{ has an even number of } a\text{'s and an odd number of } b\text{'s}\}$

(d) $L_1 \cap L_3$
Conclusion

Any other questions?