10. Randomized Algorithms: color coding and monotone local search

COMP6741: Parameterized and Exact Computation

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Contents

1 Introduction

Random Algorithms

- Turing machines do not inherently have access to randomness.
- Assume algorithm is also given access apart to a stream of *random bits*.
- With r random bits, the probability space is the set of all 2^r possible strings of random bits (with uniform distribution).

Monte Carlo algorithms

Definition 1.
- A *Monte Carlo algorithm* is an algorithm whose output is incorrect with probability at most p.
- A *one sided* error means that an algorithm's input is incorrect only on true outputs, or false outputs but not both.
- A *false negative* Monte Carlo algorithm is always correct when it returns false.

Suppose we have an algorithm A for a decision problem which:

- If no-instance: returns "no".
- If yes-instance: returns "yes" with probability p.

Algorithm A is a *one-sided Monte Carlo algorithm with false negatives*.

Problem
Suppose A is a one-sided Monte Carlo algorithm with false negatives, that with probability p returns "yes" when the input is a yes-instance. How can we use A and design an a new algorithm which ensures a new success probability of a constant C?

Amplification

Theorem 2. *If a one-sided error Monte Carlo Algorithm has success probability at least p, then repeating it independently* $\lceil \frac{1}{p} \rceil$ *times gives constant success probability. In particular if* $p = \frac{1}{f(k)}$ *for some computable function* f*, then we get an FPT one-sided error Monte Carlo Algorithm with additional* $f(k)$ *overhead in the running time bound.*

2 Vertex Cover

For a graph $G = (V, E)$ a *vertex cover* $X \subseteq V$ is a set of vertices such that every edge is adjacent to a vertex in X.

Vertex Cover	
Input:	Graph G, integer k
Parameter:	k
Question:	Does G have a vertex cover of size k?

Theorem 3. *There exists a randomized algorithm that, given a* Vertex Cover *instance* (G, k)*, in time* $2^k n^{O(1)}$ *either reports a failure or finds a vertex cover on* k *vertices in* G*. Moreover, if the algorithm is given a yes-instance, it returns a solution with constant probability.*

3 Feedback Vertex Set

A *feedback vertex set* of a multigraph $G = (V, E)$ is a set of vertices $S \subset V$ such that $G - S$ is acyclic.

Feedback Vertex Set	
Input:	Multigraph G, integer k
Parameter:	k
Question:	Does G have a feedback vertex of size k?

- Recall 5 simplification rules for Feedback Vertex Set.

Lemma 4. *Let* G *be a multigraph on* n *vertices, with minimum degree at least 3. Then, for every feedback vertex set* X *of* G*, at least 1/3 of the edges have at least one end point in* X*.*

Random Algorithm

Theorem 5. *There is a randomized algorithm that, given a Feedback Vertex Set instance* (G, k)*, in time* $6^k n^{O(1)}$ *either reports a failure or finds a feedback vertex set in* G *of at most* k*. Moreover, if the algorithm is given a yes-instance, it returns a solution with constant probability.*

Lemma 6. *Let* G *be a multigraph on* n *vertices, with minimum degree 3. For every feedback vertex set* X*, then at least* $\frac{1}{2}$ *of the edges of* G *have at least one endpoint in* X*.*

Hint: Let $H = G - X$ be a forest. The statement is equivalent to:

$$|E(G) \backslash E(H)| > |E(G)| > |V(H)|$$

Let $J \subseteq E(G)$ denote edges with one endpoint in X, and the other in $V(H)$. Show:

$$|J| > |V(H)|$$

Random Algorithm 2

Lemma 7. *There exists a randomized algorithm that, given a* Feedback Vertex Set *instance* (G, k)*, in time* $4^k n^{O(1)}$ *either reports a failure or finds a path on* k *vertices in* G*. Moreover, if the algorithm is given a yes-instance, it returns a solution with constant probability.*

Corollary 8. *Given a Feedback Vertex Set instance* (G, k)*, in time* $4^k n^{O(1)}$ *there is an algorithm that either reports a failure or if given a yes-instance finds a feedback vertex set in* G *of size at most* k *with constant probability.*

4 Color Coding

Longest Path

A *simple path* is a sequence of edges which connect a sequence of distinct vertices.

LONGEST PATH	
Input:	Graph G, integer k
Parameter:	k
Question:	Does G have a simple path of size k?

Problem

- Show that LONGEST PATH is NP-hard.

Color Coding

Lemma 9. *Let U be a set of size n, and let $X \subseteq U$ be a subset of size k. Let $\chi : U \to [k]$ be a coloring of the elements of U, chosen uniformly at random. Then the probability that the elements of X are colored with pairwise distinct colors is at least e^{-k}.*

Colorful Path

A path is *colorful* if all vertices of the path are colored with pairwise distinct colors.

Lemma 10. *Let G be an undirected graph, and let $\chi : V(G) \to [k]$ be a coloring of its vertices with k colors. There exists a determinisitic algorithm that checks in time $2^k n^{\mathcal{O}(1)}$ whether G contains a colorful path on k vertices and, if this is the case, returns one such path.*

Longest Path

Theorem 11. *There exists a randomized algorithm that, given a* LONGEST PATH *instance (G, k), in time $(2e)^k n^{O(1)}$ either reports a failure or finds a path on k vertices in G. Moreover, if the algorithm is given a yes-instance, it returns a solution with constant probability.*

5 Monotone Local Search

Exact Exponential Algorithms vs Parameterized Algorithms

Exact Exponential Algorithms

- Find exact solutions with respect to parameter n, the input size.
- Feedback Vertex set $O(1.7347^n)$ [Fomin, Todinca and Villanger 2015]
- Running Time: $O(\alpha^n n^{O(1)})$

Parameterized Algorithms

- Include parameter k, commonly the solution size.
- Feedback Vertex Set: $O(3.592^k)$ [Kociumaka and Pilipczuk 2013]
- Running Time: $O(f(k) \cdot n^{O(1)})$

Can we use Parameterized Algorithms to design fast Exact Exponential Algorithms?

Subset Problems

An *implicit set system* is a function Φ with:

- Input: instance $I \in \{0,1\}^*$, $|I| = N$
- Output: set system $(U_I, \mathcal{F}_I)$:
  - universe U_I, $|U_I| = n$
  - family $\mathcal{F}_I$ of subsets of U_I

Φ-Subset	
Input:	Instance I
Question:	Is $\lvert\mathcal{F}_I\rvert > 0$

Φ-Extension	
Input:	Instance I, a set $X \subseteq U_I$, and an integer k
Question:	Does there exist a subset $S \subseteq (U_I \backslash X)$ such that $S \cup X \in \mathcal{F}_I$ and $\lvert S\rvert \leq k$?

Algorithm

Suppose Φ-Extension has a $O^*(c^k)$ time algorithm B.

Algorithm for checking whether contains a set of size k

- Set $t = \max\left(0, \frac{ck-n}{c-1}\right)$
- Uniformly at random select a subset $X \subseteq U_I$ of size t
- Run $B(I, X, k-t)$

Running time: [Fomin, Gaspers, Lokshtanov & Saurabh 2016]

$$O^*\left(\frac{\binom{n}{t}}{\binom{k}{t}} \cdot c^{k-t}\right) = O^*\left(2 - \frac{1}{c}\right)^n$$

Intuition

Brute-force randomized algorithm

- Pick k elements of the universe one-by-one.
- Suppose $\mathcal{F}_I$ contains a set of size k.

Success probability:

$$\frac{k}{n} \cdot \frac{k-1}{n-1} \cdot \ldots \cdot \underbrace{\frac{k-t}{n-t}}_{\substack{\parallel \\ \frac{1}{c}}} \cdot \ldots \cdot \frac{2}{n-(k-2)} \frac{1}{n-(k-1)} = \frac{1}{\binom{n}{k}}$$

Theorem 12. *If there exists an algorithm for Φ-Extension with running time $c^k n^{O(1)}$ then there exists a randomized algorithm for Φ-Subset with running time $(2 - \frac{1}{c})^n \cdot n^{O(1)}$*

- Can be derandomized at the expense of a multiplicative $2^{o(1)}$ factor in the running time.

Theorem 13. *For a graph G there exists a randomized algorithm which finds a smallest feedback vertex set in time $\left(2 - \frac{1}{3.592}\right)^n \cdot n^{O(1)} = 1.7217^n \cdot n^{O(1)}$.*

References

- Chapter 5, *Randomized methods in parameterized algorithms* by Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. Parameterized Algorithms. Springer, 2015.
- *Exact Algorithms via Monotone Local Search*, Fedor V. Fomin, Serge Gaspers, Daniel Lokshtanov, Saket Saurabh. ACM symposium on Theory of Computing, 2016.

Bonus Slides 1

1-Regular Deletion	
Input:	Graph $G = (V, E)$, integer k
Parameter:	k
Question:	Does there exist $X \subseteq V$ with $\|X\| \leq k$ such that $G - X$ is 1-regular?

- Design a randomized FPT algorithm with running time $O^*(4^k)$

Bonus Slides 2

Triangle Packing	
Input:	Graph G, integer k
Parameter:	k
Question:	Does G have k-vertex disjoint triangles?

- Design a randomized FPT algorithm for Triangle Packing.