Exercise 1. A *domatic k-partition* of a graph $G = (V, E)$ is a partition $(D_1, \ldots, D_k)$ of $V$ into $k$ dominating sets of $G$.

<table>
<thead>
<tr>
<th>(sol+tw)-DOMATIC PARTITION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input: graph $G$, integer $k$</td>
</tr>
<tr>
<td>Parameter: $k + \text{tw}(G)$</td>
</tr>
<tr>
<td>Question: Does $G$ have a domatic $k$-partition.</td>
</tr>
</tbody>
</table>

- Show that (sol+tw)-DOMATIC PARTITION is FPT using Courcelle's theorem

**Solution.** To show that (sol+tw)-DOMATIC PARTITION is FPT, we express it as an MSO sentence which is true for the input graph $G$ if and only if $G$ has a domatic $k$-partition:

$$\exists D_1 \subseteq V \exists D_2 \subseteq V \ldots \exists D_k \subseteq V \quad \text{partition}(D_1, D_2, \ldots, D_k) \wedge$$
$$\forall v \in V \ dom(v, D_1) \wedge \cdots \wedge dom(v, D_k)$$

with

$$\text{partition}(D_1, \ldots, D_k) := \forall v \in V \ (v \in D_1 \wedge v \notin D_2 \wedge v \notin D_3 \wedge \cdots \wedge v \notin D_k) \vee$$
$$\ (v \notin D_1 \wedge v \in D_2 \wedge v \notin D_3 \wedge \cdots \wedge v \notin D_k) \vee$$
$$\cdots$$
$$\ (v \notin D_1 \wedge v \notin D_2 \wedge v \in D_3 \wedge \cdots \wedge v \notin D_k)$$

and

$$\text{dom}(v, X) := v \in X \vee \exists x \in X \ adj(v, w)$$

The length of this expression is $O(k^2)$. Since this is a parameterized reduction to Courcelle’s problem, the result follows.

Exercise 2. Show that the incidence treewidth of a CNF formula $F$ is at most the dual treewidth of $F$ plus 1.

**Solution.** Start from a tree decomposition $(T, \gamma)$ of the dual graph of $F$ with minimum width. For each variable $v$ in $F$, select a bag $i_v$ that contains all the clauses where $v$ occurs. Such a bag necessarily exists, since these clauses form a clique in the dual graph. Add a new bag containing $v$ and all the clauses where $v$ occurs, and make this bag adjacent to $i_v$. This gives a tree decomposition for the incidence graph of $F$ whose width equals the width of the tree decomposition of the dual graph plus one.

Exercise 3. Show that CSP is W[1]-hard for parameter incidence treewidth and Boolean domain ($D = \{0, 1\}$).

**Hints.** Reduce from CLIQUE.

1. Use Boolean variables $x_{ij}$ with $1 \leq i \leq k$ and $1 \leq j \leq n$ with the meaning that $x_{ij}$ is set to 1 if the $i$th vertex of the clique corresponds to the $j$th vertex in the graph.
2. Add $O(k^2)$ constraints enforcing that for each $i \in \{1, \ldots, k\}$, exactly one $x_{ij}$ is set to 1, and whenever two $x_{ij}, x_{ij'}$ with $i \neq i'$ are set to 1, then vertices $j$ and $j'$ are adjacent in the graph.
3. Show that a graph with a vertex cover of size $q$ has treewidth at most $q$.

Exercise 4. Design an $O^*(2^k)$ time DP algorithm for tw-INDEPENDENT SET.
**tw-Independent Set**

<table>
<thead>
<tr>
<th>Input:</th>
<th>Graph G, integer k, and a tree decomposition of G of width t</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter:</td>
<td>t</td>
</tr>
<tr>
<td>Question:</td>
<td>Does G have an independent set of size k?</td>
</tr>
</tbody>
</table>

**Solution sketch.**

- Obtain a nice tree decomposition \((T, \gamma)\) of width \(t\) in polynomial time.
- Denote \(T_i\) the subtree of \(T\) rooted at node \(i\).
- Denote \(\gamma_i(i) = \{v \in \gamma(j) : j \in V(T_i)\}\) and \(G_i(i) = G[\gamma_i(i)]\).
- For each node \(i\) of \(T\), and each \(S \subseteq \gamma(i)\), compute \(\text{ind}(i, S)\), the size of a largest independent set of \(G_i(i)\) that contains all vertices of \(S\) and no vertex from \(\gamma(i) \setminus S\) by dynamic programming.
- For a leaf node \(i\) with \(\gamma(i) = \{v\}\):
  \[
  \text{ind}(i, \emptyset) = 0 \\
  \text{ind}(i, \{v\}) = 1
  \]
- For a forget node \(i\) with child \(i'\) and \(\gamma(i) = \gamma(i') \setminus \{v\}\):
  \[
  \text{ind}(i, S) = \max(\text{ind}(i', S), \text{ind}(i', S \cup \{v\}))
  \]
- For an introduce node \(i\) with child \(i'\) and \(\gamma(i) = \gamma(i') \cup \{v\}\):
  \[
  \text{ind}(i, S) = \begin{cases} 
  -\infty & \text{if } G[S] \text{ contains an edge} \\
  \text{ind}(i', S \setminus \{v\}) + 1 & \text{if } v \in S \\
  \text{ind}(i', S \cup \{v\}) & \text{otherwise}
  \end{cases}
  \]
- For a join node \(i\) with children \(i'\) and \(i''\):
  \[
  \text{ind}(i, S) = \text{ind}(i', S) + \text{ind}(i'', S) - |S|
  \]

**Exercise 5.** Design an \(O^*(9^t)\) time DP algorithm for \(\text{tw-Dominating Set}\). Can you even achieve an \(O^*(4^t)\) time DP algorithm?

**tw-Dominating Set**

<table>
<thead>
<tr>
<th>Input:</th>
<th>Graph G, integer k, and a tree decomposition of G of width at most t</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter:</td>
<td>t</td>
</tr>
<tr>
<td>Question:</td>
<td>Does G have a dominating set of size k?</td>
</tr>
</tbody>
</table>

**Solution sketch.**

- Obtain a nice tree decomposition \((T, \gamma)\) of width \(t\) in polynomial time.
- Denote \(T_i\) the subtree of \(T\) rooted at node \(i\).
- Denote \(\gamma_i(i) = \{v \in \gamma(j) : j \in V(T_i)\}\)
- Denote \(G_i(i) = G[\gamma_i(i)]\)
- For each node \(i\) of \(T\), and each labelling \(\ell : \gamma(i) \rightarrow \{\text{in}, \text{outDom}, \text{outNd}\}\), compute the smallest size of a subset \(D\) of \(\gamma_i(i)\) such that \(D \cap \gamma(i)\) is the set of vertices labelled \(\text{in}\) by \(\ell\), and that dominates all vertices from \(\gamma_i(i)\) except those that are labeled \(\text{outNd}\) by \(\ell\) by dynamic programming.

The running time depends on how join nodes are handled. See Section 10.5 in the [Niedermeier, '06] textbook for details.