2. Dynamic Programming
COMP6741: Parameterized and Exact Computation

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Outline

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1. Dynamic Programming Across Subsets
   - Traveling Salesman Problem
   - Coloring
   - Dominating Set in bipartite graphs

2. Further Reading

Dynamic Programming across Subsets

- very general technique
- uses solutions of subproblems
- typically stored in a table of exponential size

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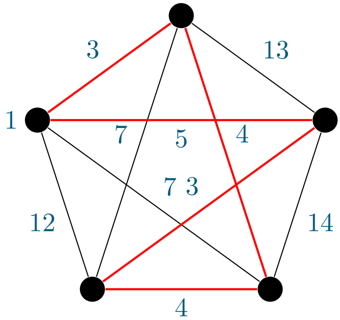
Traveling Salesman Problem (TSP)

Input: a set of n cities, the distance $d(i, j) \in \mathbb{N}$ between every two cities i and j, integer k

Question: Is there a permutation of the cities (a tour) such that the total distance when traveling from city to city in the specified order, and returning back to the origin, is at most k?



Brute-force: Try all permutations of cities; $O^*(n!)$

Dynamic Programming for TSP I

For a non-empty subset of cities $S \subseteq \{2, 3, ..., n\}$ and city $i \in S$:

- $\textsc{Opt}[S; i] \equiv$ length of the shortest path starting in city 1, visits all cities in $S \setminus \{i\}$ and ends in i.

Then,

$$\begin{aligned}\textsc{Opt}[\{i\}; i] &= d(1, i) \\ \textsc{Opt}[S; i] &= \min\{\textsc{Opt}[S \setminus \{i\}; j] + d(j, i) : j \in S \setminus \{i\}\}\end{aligned}$$

- For each subset S in order of increasing cardinality, compute $\textsc{Opt}[S; i]$ for each i.
- Final solution:

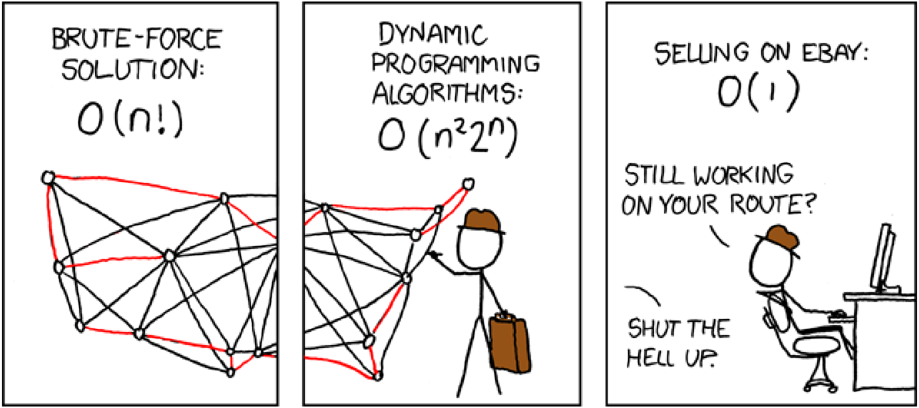
$$\min_{2 \le j \le n} \{\textsc{Opt}[\{2, 3, ..., n\}; j] + d(j, 1)\}$$

Dynamic Programming for TSP II

Theorem 1 (Held & Karp '62)

TSP can be solved in time $O(2^n n^2) = O^*(2^n)$.

- best known algo for TSP


BRUTE-FORCE SOLUTION:
O(n!)
DYNAMIC PROGRAMMING ALGORITHMS:
O(n²2ⁿ)
SELLING ON EBAY:
O(1)
STILL WORKING ON YOUR ROUTE?
SHUT THE HELL UP.

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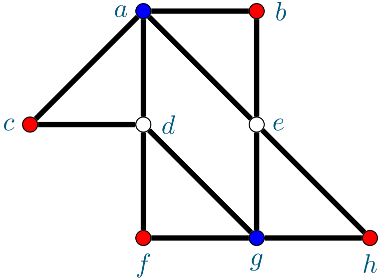
Coloring

A k-coloring of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, ..., k\}$ assigning colors to V such that no two adjacent vertices receive the same color.

Coloring	
Input:	Graph G, integer k
Question:	Does G have a k-coloring?



Exercise

A k-coloring of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, ..., k\}$ assigning colors to V such that no two adjacent vertices receive the same color.

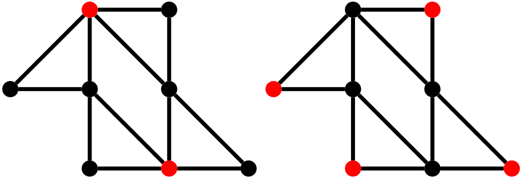
COLORING	
Input:	Graph G, integer k
Question:	Does G have a k-coloring?

Exercise

Design an $O^*(4^n)$ time algorithm for COLORING.

Maximal Independent Sets

- An independent set is maximal if it is not a subset of any other independent set.
- Examples:



Coloring and Maximal Independent Sets

Theorem 2 ([Moon, Moser '65], [Johnson, Yannakakis, Papadimitriou '88])

A graph on n vertices contains at most $3^{n/3} \subseteq O(1.4423^n)$ maximal independent sets. Moreover, they can all be enumerated in time $O^*(3^{n/3})$.

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Lemma 3 ([Lawler '76])

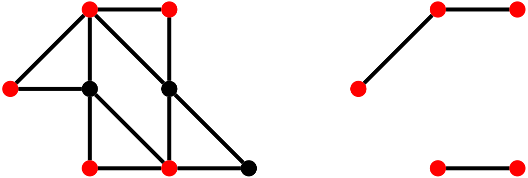
For any graph G, there exists an optimal coloring for G where one color class is a maximal independent set in G.

Proof.

Exercise □

Dynamic Programming for $\textsc{Coloring}$ I

- $G[S] \equiv$ subgraph of G induced by the vertices in S



- $\textsc{Opt}[S] \equiv$ minimum k such that $G[S]$ is k-colorable.
- Then,

$$
\begin{aligned}
\textsc{Opt}[\emptyset] &= 0 \\
\textsc{Opt}[S] &= 1 + \min\{\textsc{Opt}[S \setminus I] : I \text{ maximal ind. set in } G[S]\}
\end{aligned}
$$

Dynamic Programming for Coloring II

$$\begin{aligned} \textsc{Opt}[\emptyset] &= 0 \\ \textsc{Opt}[S] &= 1 + \min\{\textsc{Opt}[S \setminus I] : I \text{ maximal ind. set in } G[S]\} \end{aligned}$$

- go through the sets S in order of increasing cardinality
- to compute $\textsc{Opt}[S]$, generate all maximal independent sets I of $G[S]$
- this can be done in time $|S|^2 3^{|S|/3}$
- time complexity:

$$\sum_{s=0}^{n} \binom{n}{s} s^2 3^{s/3} \le n^2 \sum_{s=0}^{n} \binom{n}{s} 3^{s/3} = n^2 (1 + 3^{1/3})^n = O(2.4423^n)$$

[Recall the Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.]

Dynamic Programming for Coloring III

Theorem 4 ([Lawler '76])

Coloring *can be solved in time* $O(2.4423^n)$.

- was best known algorithm for 25 years (until [Eppstein '01])
- current best: $O^*(2^n)$ [Bjørklund & Husfeldt '06], [Koivisto '06]

k-Coloring for small k

k-COLORING
Input: Graph G, integer k
Question: Does G have a k-coloring?

- $k \leq 2$: polynomial
- $k > 2$: NP-complete

Algorithm for 3-Coloring

Theorem 5 ([Lawler '76])

3-COLORING *can be decided in time* $O(1.4423^n)$.

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Proof.

For every maximal independent I set of G, check if $G - I$ is 2-colorable. □

Algorithm for 3-Coloring

Theorem 5 ([Lawler '76])

3-COLORING *can be decided in time* $O(1.4423^n)$.

Proof.

For every maximal independent I set of G, check if $G - I$ is 2-colorable. $\square$

current best: $O(1.3289^n)$ [Eppstein '01]

Algorithm for 4-Coloring

Theorem 6

4-Coloring *can be decided in time* $O(1.7851^n)$.

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Proof.

- For each maximal independent set I of G of size at least $n/4$, check if $G - I$ is 3-colorable.

Algorithm for 4-Coloring

Theorem 6

4-COLORING *can be decided in time* $O(1.7851^n)$.

Proof.

- For each maximal independent set I of G of size at least $n/4$, check if $G - I$ is 3-colorable.
- We need to prove that each 4-colorable graph G has a 4-coloring where one color class is a maximal i.s. of size $\geq n/4$?
- Pick a 4-coloring of G.
- ≥ 1 color class contains $\geq n/4$ vertices.
- If this color class is not a maximal i.s., recolor some other vertices such that it becomes a maximal i.s.

Algorithm for 4-Coloring

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- For each maximal independent set I of G of size at least $n/4$, check if $G - I$ is 3-colorable.
- We need to prove that each 4-colorable graph G has a 4-coloring where one color class is a maximal i.s. of size $\geq n/4$?
- Pick a 4-coloring of G.
- ≥ 1 color class contains $\geq n/4$ vertices.
- If this color class is not a maximal i.s., recolor some other vertices such that it becomes a maximal i.s.
- Running time: $O(3^{n/3} 1.3289^{3n/4}) \subseteq O(1.7851^n)$

□

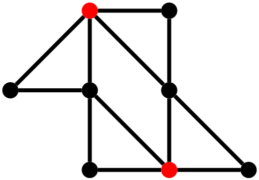
current best: $O(1.7272^n)$ [Fomin, Gaspers, Saurabh '07]

Outline

Dominating Set

A dominating set in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that each vertex of G is either in S or adjacent to a vertex in S.

Dominating Set	
Input:	Graph G, integer k
Question:	Does G have a dominating set of size k?



Bipartite graphs

A graph $G = (V, E)$ is bipartite if its vertex set can be partitioned into two independent sets.

Dominating Set in Bipartite Graphs	
Input:	Bipartite graph G, integer k
Question:	Does G have a dominating set of size k?

Note: Dominating Set in Bipartite Graphs is NP-complete.

Algorithm for Dominating Set in Bipartite Graphs I

Partition V into independent sets A and B, with $|B| \geq |A|$.

The algorithm has 2 phases:

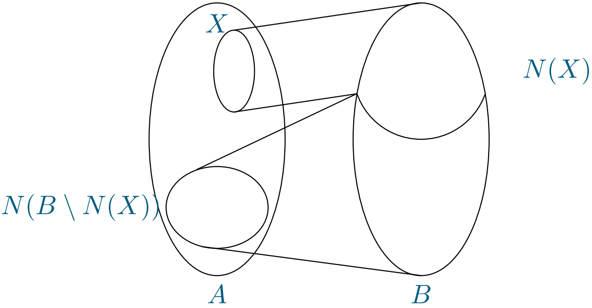
- **Preprocessing phase**: compute for each $X \subseteq A$ a subset $\text{Opt}[X]$ which is a smallest subset of B that dominates X.
- **Main phase**: for each subset $X \subseteq A$, compute a dominating set D of G of minimum size such that $D \cap A = X$.

Algorithm for Dominating Set in Bipartite Graphs II

Main phase. For a vertex subset $X \subseteq A$, a dominating set D of G of minimum size such that $D \cap A = X$ is obtained by setting

$$D := X \cup (B \setminus N(X)) \cup \mathsf{Opt}[A \setminus (X \cup N(B \setminus N(X)))]$$

if $A \setminus X$ contains no degree-0 vertex.
(If $A \setminus X$ contains a degree-0 vertex, we skip this set X, because there is no dominating set D of G with $D \cap A = X$.)



Algorithm for Dominating Set in Bipartite Graphs III

Preprocessing phase. Let $B = \{b_1, \ldots, b_{|B|}\}$. We compute for each $X \subseteq A$ and integer k, $0 \leq k \leq |B|$, a subset $\mathsf{Opt}[X, k] \subseteq \{b_1, \ldots, b_k\}$ which is defined as

- a smallest subset of $\{b_1, \ldots, b_k\}$ that dominates X if $X \subseteq N(\{b_1, \ldots, b_k\})$, and
- B if $X \not\subseteq N(\{b_1, \ldots, b_k\})$.

Note: $\mathsf{Opt}[X, |B|] = \mathsf{Opt}[X]$.

Algorithm for Dominating Set in Bipartite Graphs IV

Base cases

$$\begin{aligned} \mathsf{Opt}[\emptyset, k] &= \emptyset & \forall k \in \{0, \dots, |B|\}, \\ \mathsf{Opt}[X, 0] &= B & \forall X,\ \emptyset \subsetneq X \subseteq A. \end{aligned}$$

Dynamic Programming recurrence

$$\mathsf{Opt}[X, k] = \begin{cases} \mathsf{Opt}[X, k-1] & \text{if } |\mathsf{Opt}[X, k-1]| < 1 + |\mathsf{Opt}[X \setminus N(b_k), k-1]| \\ \{b_k\} \cup \mathsf{Opt}[X \setminus N(b_k), k-1] & \text{otherwise} \end{cases}$$

for each $X,\ \emptyset \subsetneq X \subseteq A$ and $k \in \{1, \dots, |B|\}$.

Algorithm for Dominating Set in Bipartite Graphs V

Theorem 7 ([Liedloff '08])

DOMINATING SET IN BIPARTITE GRAPHS *can be solved in* $O^*(2^{n/2})$ *time, where* n *is the number of vertices of the input graph.*

Outline

Further Reading

- Chapter 3, *Dynamic Programming* in
  Fedor V. Fomin and Dieter Kratsch. Exact Exponential Algorithms. Springer, 2010.