Exercise 1. Show that Path Packing has no polynomial kernel unless NP ⊆ coNP/poly.

**Solution.** We give a polynomial parameter transformation from Long Path to Path Packing. Given an instance \((G, k)\) to Long Path we construct a graph \(G'\) from \(G\) by adding \(k - 1\) vertex-disjoint paths of length \(k\). Now, \(G\) contains a path of length \(k\) if and only if \(G'\) contains \(k\) vertex-disjoint paths of length \(k\). The construction takes \(O(k \cdot (n+m))\) time, where \(n\) and \(m\) are the number of vertices and edges of \(G\), respectively, and the target parameter is \(k\).

Exercise 2. An endpoint of a path is a vertex that has degree at most 1 in the path. Consider the NP-complete Anchored Path problem.

**Solution.** We give an OR-composition and use the Composition Theorem. To use the Composition Theorem, we need that Anchored Path is

- NP-complete (given in the question statement),
- the parameter can be computed in polynomial time (given in the input),
- and the value of the parameter is at most the instance size (guaranteed in the input specification).

Consider a finite sequence of instances \(((G_i, r_i, k))_{1 \leq i \leq t}\) for Anchored Path.

**Construction.** Our OR-composition algorithm computes an instance \((G, r, k + 1)\) that is obtained by taking the disjoint union of all \(G_i\), \(1 \leq i \leq t\), adding a new vertex \(r\) that is adjacent to all \(r_i\), \(1 \leq i \leq t\).

**Correctness.** If some \(G_i\) has a path of length \(k\) starting from \(r_i\), then adding \(r\) to the beginning of that path gives a path of length \(k + 1\) in \(G\) starting from \(r\). On the other hand, if \(G\) has a path of length \(k + 1\) starting from \(r\), the second vertex in that path is some \(r_i\), and the following \(k - 1\) vertices of the path are all in \(G_i\), and so \(G_i\) has a path of length \(k\) starting from \(r_i\).

**Running time.** Clearly polynomial in the input size.