1 Review

1.1 Upper Bounds

Kernelization: definition

Definition 1. A kernelization for a parameterized problem Π is a polynomial time algorithm, which, for any instance I of Π with parameter k, produces an equivalent instance I' of Π with parameter k' such that |I'| ≤ f(k) and k' ≤ f(k) for a computable function f. We refer to the function f as the size of the kernel.

Main Complexity Classes

P: class of problems that can be solved in time \( n^{O(1)} \)

FPT: class of problems that can be solved in time \( f(k) \cdot n^{O(1)} \)

W[1]: parameterized intractability classes

XP: class of problems that can be solved in time \( f(k) \cdot n^{o(k)} \)

\[ P \subseteq \text{FPT} \subseteq W[1] \subseteq W[2] \ldots \subseteq W[P] \subseteq \text{XP} \]

Known: If FPT = W[1], then the Exponential Time Hypothesis fails, i.e. 3-SAT can be solved in time \( 2^{o(n)} \).

Search trees
Recall: A search tree models the recursive calls of an algorithm. For a b-way branching where the parameter k decreases by a at each recursive call, the number of nodes is at most \( b^{k/a} \cdot (k/a + 1) \).

If k/a and b are upper bounded by a function of k, and the time spent at each node is FPT (typically, polynomial), then we get an FPT running time.
Measure & Conquer

Lemma 2 (Measure & Conquer Lemma). Let

- A be a branching algorithm
- \( c \geq 0 \) be a constant, and
- \( \mu(\cdot), \eta(\cdot) \) be two measures for the instances of A,

such that on input I, A calls itself recursively on instances \( I_1, \ldots, I_k \), but, besides the recursive calls, uses time \( O(|I|^c) \), such that

\[
(\forall i) \quad \eta(I_i) \leq \eta(I) - 1, \quad \text{and} \quad 2^\mu(I_1) + \ldots + 2^\mu(I_k) \leq 2^\mu(I).
\]

Then A solves any instance I in time \( O(\eta(I)^{c+1}) \cdot 2^\mu(I). \)

Tree decompositions (by example)

- A graph \( G \)

```
  a — b — c — d
     \   \   \   
     e — f — g — i — j
        \   \   \   
        h — f — i — k
```

- A tree decomposition of \( G \)

```
  a, b, c — c, d, e — d, e, f — d, f, h — h, i — i, j, k
        \   \   \   \   \   
        f, g
```

Conditions: covering and connectedness.

Randomized algorithms

Solution intersects a linear number of edges:

- Sampling vertices with probability proportional to their degree gives good success probability if the set of vertices we try to find has large intersection with the edges of the graph.

Color Coding:

Lemma 3. Let \( X \subseteq U \) be a subset of size \( k \) of a ground set \( U \). Let \( \chi : U \rightarrow \{1, \ldots, k\} \) be a random coloring of \( U \). The probability that the elements of \( X \) are colored with pairwise distinct colors is \( \geq e^k \).

Monotone Local Search:

- For many subset problems a \( O^*(c^k) \) algorithm for finding a solution of size \( k \) can be turned into a randomized algorithm finding an optimal solution in time \( O^*((2 - 1/c)^n) \).
1.2 Lower Bounds

Reductions
We have seen several reductions, which, for an instance \((I,k)\) of a problem \(\Pi\), produce an equivalent instance \(I'\) of a problem \(\Pi'\).

<table>
<thead>
<tr>
<th>Parameterized reduction</th>
<th>time parameter</th>
<th>special features</th>
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<td>polynomial parameter transformation</td>
<td>poly</td>
<td>(k' \leq \text{poly}(k))</td>
<td>(Kernel LBs)</td>
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<td>SubExponential Reduction Family</td>
<td>subexp((k))</td>
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2 Research in Parameterized and Exact Computation

News
- Recently solved open problems from [DF13]
  - Biclique is W[1]-hard [Lin18]
  - Short Generalized Hex is W[1]-complete [Bon+17]
  - Determining the winner of a Parity Game is FPT in the number of values [Cal+17]
- Research focii
  - Enumeration algorithms and combinatorial bounds
  - Randomized algorithms
  - Treewidth: computation, bounds
  - Bidimensionality
  - Bottom-up: improving the quality of subroutines of heuristics
  - (S)ETH widely used now, also for poly-time lower bounds
  - FPT-approximation algorithms, lossy kernels
  - General-purpose “modeling” problems: SAT, CSP, ILP, Integer Quadratic Programming
  - Backdoors

Resources
- FPT wiki: [http://fpt.wikidot.com](http://fpt.wikidot.com)
- cstheory stackexchange: [http://cstheory.stackexchange.com](http://cstheory.stackexchange.com)
- FPT summer schools (include lecture slides)
  - 2017: [https://algo2017.ac.tuwien.ac.at/pcss/](https://algo2017.ac.tuwien.ac.at/pcss/)
  - 2014: [http://fptschool.mimuw.edu.pl](http://fptschool.mimuw.edu.pl)
References


