Updating belief

1. Bayesian updating
   - Airline case study

2. Value of information

3. Revision of Bayesian beliefs
   - Incorporating additional information
   - Updating reliability likelihood

4. Sensitivity analysis
Bayesian updating

Updating belief

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Case study: capital purchase

Example (To purchase or not)

You’re the chief engineer of a small commercial airline which, due to increased demand, is considering adding to its fleet by buying (B) a used airliner. Another company is offering to sell one of its airliners for $400,000. Used airlines range in reliability, which is hard to evaluate without a detailed inspection.

Question: should you purchase?
Problem modelling

- Problem 1: how to measure reliability? Operating hours
- Simplification 1: classify airliners as either: very reliable \((vR) (> 90\%)\), moderately reliable \((mR)\), or unreliable \((uR) (< 50\%)\)

Beliefs about reliability:

<table>
<thead>
<tr>
<th>Reliability</th>
<th>(vR)</th>
<th>(mR)</th>
<th>(uR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.2</td>
<td>0.3</td>
<td>0.5</td>
</tr>
<tr>
<td>Utility</td>
<td>1.0</td>
<td>0.34</td>
<td>0.01</td>
</tr>
</tbody>
</table>

- Simplification 2: assume a very reliable airliner makes $1M profit (best outcome); an unreliable one makes $200K loss (worst)
- Simplification 3: utility of not buying airliner—*status quo*: 0.17

Decision C (buy or not)

Buy:

\[
U(B) = 0.2(1.0) + 0.3(0.34) + 0.5(0.01) = 0.31
\]

\[
U(\bar{B}) = 0.17
\]
Decision C

- Evaluate decision points/nodes by maximal utility of alternatives (i.e., actions/strategies)
- The value of node $C$ is $0.31$, because $0.31 > 0.17$; i.e., $0.31 = \max\{0.17, 0.31\}$

$0.31 = \max\{0.17, 0.31\}$

Value of information

Updating belief
Get more information?

Example (Additional information)

You have the option to consult an aeronautical engineering firm to conduct an assessment of the airliner for $10K. The report will be either favourable ($f$) or unfavourable ($u$) as to whether or not to purchase.

- Firm’s assessment reliable?
- Records: 90% of very reliable planes receive favourable assessment; i.e., $P(f|vR) = 0.9$

<table>
<thead>
<tr>
<th>Probability of:</th>
<th>$vR$</th>
<th>$mR$</th>
<th>$uR$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td>0.9</td>
<td>0.6</td>
<td>0.1</td>
</tr>
<tr>
<td>$u$</td>
<td>0.1</td>
<td>0.4</td>
<td>0.9</td>
</tr>
</tbody>
</table>

Revision of Bayesian beliefs

Updating belief

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4. Sensitivity analysis
Conditional probability

Definition
The \textit{conditional probability} of event $A$ conditional on $B$ (provided $B$ is possible; \textit{i.e.}, $P(B) \neq 0$), written $P(A|B)$, is defined by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

In the diagram above, $P(A|B)$ represents the ratio of (the area of) the region $AB$ (the dark region) to that of the whole of $B$.

Conditional independence

Definition
Event $A$ is \textit{(conditionally) independent} of event $B$ if:

$$P(A|B) = P(A).$$

Event $A$ is \textit{(conditionally) dependent} on $B$ if $A$ is not (conditionally) independent of $B$.

For example, if $B$ is a random sample of a population.
Bayes’s rule

- Rearranging the definition of conditional probability:

\[ P(A \cap B) = P(A|B)P(B) \]

- By symmetry \( P(A \cap B) = P(B \cap A) \); therefore:

\[ P(A|B)P(B) = P(B|A)P(A) \]

Rearranging gives:

**Theorem (Bayes's Theorem I)**

*If \( A \) and \( B \) are any two events (such that \( P(A) \neq 0 \)), then:*

\[ P(B|A) = \frac{P(A|B)P(B)}{P(A)} \]

Bayes’s Venn diagram

But \( A = AB \cup A\overline{B} \). So we get the following:

**Theorem (Bayes’s Theorem I')**

*If \( A \) and \( B \) are any two events (\( P(A) \neq 0 \)), then:*

\[ P(B|A) = \frac{P(AB)}{P(AB) + P(A\overline{B})} \]
Revise of Bayesian beliefs  
Incorporating additional information

**Extending Bayes’s rule**

![Diagram](image)

\[
P(B_1|A) = \frac{P(AB_1)}{P(AB_1) + P(AB_2) + P(AB_3)}
\]

\[
P(B_2|A) = \frac{P(AB_2)}{P(AB_1) + P(AB_2) + P(AB_3)}
\]

\[
P(B_3|A) = \frac{P(AB_3)}{P(AB_1) + P(AB_2) + P(AB_3)}
\]

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Revision of Bayesian beliefs  
Incorporating additional information

**Bayes’s rule generalised**

Events \(B_1, \ldots, B_n\) are said to be *universally exhaustive* (of \(\Omega\)) if \(\bigcup_{i=1}^{n} B_i = \Omega\).

**Theorem (Bayes's Theorem II)**

*If events \(B_1, \ldots, B_k, \ldots, B_n\) are mutually exclusive and universally exhaustive, and \(A\) is a possible event \((P(A) \neq 0)\), then:* 

\[
P(B_k|A) = \frac{P(AB_k)}{\sum_{i=1}^{n} P(AB_i)}
\]

**Theorem (Bayes's Theorem II’)**

*If \(B_1, \ldots, B_k, \ldots, B_n\) are mutually exclusive and universally exhaustive events and \(A\) is a possible event \((P(A) \neq 0)\) then:* 

\[
P(B_k|A) = \frac{P(A|B_k)P(B_k)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}
\]

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Example: Bayes’s rule

Example (Medical diagnostics)

In a given population of people, one in every thousand have hypo-cytocitic cancer. A certain pathology test is used to detect the disease. The test is ‘good’ but not perfect; it returns a positive result in 98% of persons with the disease, and registers a false positive (i.e., gives a positive result for a person free of the disease) 5% of the time.

Exercise

A random person comes in to get tested and the test returns a positive result. What is the probability that the person has cancer?

Example: solution

Given information:

\[
\begin{align*}
P(C) &= \frac{1}{1000} \\
P\left(T^+|C\right) &= \frac{98}{100} \\
P\left(T^+|C\right) &= \frac{5}{100} \\
P(C) &= \frac{999}{1000} \\
P\left(T^+|C\right) &= \frac{2}{100} \\
P\left(T^+|\bar{C}\right) &= \frac{95}{100}
\end{align*}
\]

What is \( P(C|T^+) \)?

\[
P(C|T^+) = \frac{P(CT^+)}{P(T^+)} = \frac{P(CT^+)}{P(T^+|C)P(C) + P(T^+|\bar{C})P(\bar{C})}
\]

\[
= \frac{\frac{98}{100} \times \frac{1}{1000} + \frac{5}{100} \times \frac{999}{1000}}{\frac{98}{100} \times \frac{1}{1000} + \frac{5}{100} \times \frac{999}{1000}} = \frac{98}{98 + 5 \times 999} \approx \frac{100}{5000} = 0.02
\]

Patient only has 2% chance of having cancer despite testing positive?!
Decision C (buy or not)

Buy:

\[
\begin{array}{c}
\text{vR : 0.2} \\
\text{mR : 0.3} \\
\text{uR : 0.5} \\
\text{0.31} \\
\text{0.34} \\
\text{0.01}
\end{array}
\]

\[
U(B) = 0.2(1.0) + 0.3(0.34) + 0.5(0.01) = 0.31
\]

\[
U(\overline{B}) = 0.17
\]

Post-report (posterior) probabilities

- If report favourable (f):
  \[
P(vR|f) = \frac{P(f|vR)P(vR)}{P(f|vR)P(vR)+P(f|mR)P(mR)+P(f|uR)P(uR)} = \frac{0.9(0.2)}{0.9(0.2) + 0.6(0.3) + 0.1(0.5)} = \frac{0.18}{0.41} \approx 0.44
  \]
  Similarly: \( P(mR|f) \approx 0.44 \) and \( P(uR|f) \approx 0.12 \)

- If report unfavourable (u):
  \[
P(vR|u) = \frac{0.02}{0.59} \approx 0.04
  \]
  \[
P(mR|u) \approx 0.20
  \]
  \[
P(uR|u) \approx 0.76
  \]
Decision A (report favourable)

The revised expected utility of buying the airliner is
\[ U(B) = 0.44(0.99) + 0.44(0.33) + 0.12(0.0) = 0.58 \]

The utility of not buying it is \( U(\overline{B}) = 0.16 \).

<table>
<thead>
<tr>
<th>vR</th>
<th>mR</th>
<th>uR</th>
<th>U</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.44</td>
<td>0.44</td>
<td>0.12</td>
<td>0.58</td>
</tr>
</tbody>
</table>

Decision B (report unfavourable)

The revised expected utility of buying the airliner is
\[ U(B) = 0.04(0.99) + 0.20(0.33) + 0.76(0.0) = 0.10 \]

The utility of not buying it is \( U(\overline{B}) = 0.16 \).

<table>
<thead>
<tr>
<th>vR</th>
<th>mR</th>
<th>uR</th>
<th>U</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>0.20</td>
<td>0.76</td>
<td>0.10</td>
</tr>
</tbody>
</table>
Utility adjustments

- Problem: cost associated with report?
  Question: How does report’s cost (\$10K) affect utility?
- Observation: report cost small relative to other monetary quantities: potential profit \$1M; i.e., \$10K \ll \$1M
- Simplification 4: model effect by constant shift; i.e., for report costing \$x (x \ll 1M), change of utility is \( \Delta u = \frac{x}{1M} \); \$1M
- That is, every \$10K is worth 0.01 utiles

Combined decision

- Combine all three possible cases into one big decision problem
- Introduce new decision: commission survey/report, and no survey/report
- Introduce new event: report outcome (f or u)
- If consultant good, report likely to be good predictor of (i.e., correlated to) aircraft reliability
- Consultant’s increased predictive accuracy is valuable in making decision
Combined decision

- From the denominators in the earlier calculations:
  \[ P(f) = 0.41 \]
  \[ P(u) = 0.59 \]

- Therefore, if report commissioned:
  \[ 0.41 \cdot 0.58 \]
  \[ 0.59 \cdot 0.16 \]

- Utility of report: 0.33

Decision table

<table>
<thead>
<tr>
<th></th>
<th>( f, vR )</th>
<th>( f, mR )</th>
<th>( f, uR )</th>
<th>( u, vR )</th>
<th>( u, mR )</th>
<th>( u, uR )</th>
<th>( U )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>1.0</td>
<td>0.34</td>
<td>0.01</td>
<td>1.0</td>
<td>0.34</td>
<td>0.01</td>
<td>0.31</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>0.17</td>
<td>0.17</td>
<td>0.17</td>
<td>0.17</td>
<td>0.17</td>
<td>0.17</td>
<td>0.17</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>0.99</td>
<td>0.33</td>
<td>0</td>
<td>0.99</td>
<td>0.33</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \cdots )</td>
</tr>
<tr>
<td>( A_6 )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
</tbody>
</table>

where

- \( A_1 \) no survey; buy airliner
- \( A_2 \) no survey; don’t buy airliner
- \( A_3 \) commission survey; buy airliner
- \( A_4 \) commission survey; don’t buy
- \( A_5 \) commission survey; if favourable, buy airliner; else don’t buy
- \( A_6 \) commission survey; if favourable, don’t buy airliner; else buy
Value of information

- Optimal policy if report commissioned:
  
  **Policy for R: report commissioned**
  
  If report favourable, buy airliner, if not don’t buy.
  
  - Value of policy is $U(R) = 0.33$, inclusive of the 0.01 fee
  - Optimal policy if report not commissioned:
    
    **Policy for $\bar{R}$: report not commissioned**
    
    Buy the airliner.
    
    - $U(\bar{R}) = 0.31$
    - How much is report worth?
    - $U(R) = 0.34 - u_r \geq 0.31 = U(\bar{R})$; i.e., should commission report for fee up to $u_r = 0.03$; i.e., for any fee up to $30K$

Sensitivity analysis

Updating belief

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4. Sensitivity analysis
Example (Production)

Alice is the CTO at a company and Bob is the CFO. They’re considering two possible production processes for a product. Process A is expected to net $40K if demand increases, $30K if demand remains stable, and $20K if demand falls. Process B requires a greater initial capital expenditure; it will only net $10K if demand drops, and $40K otherwise.

Future estimates of demand are: 20% of an increase, 30% chance of staying level, and 50% of a decrease.

Which process should Alice implement?

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>V$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$20</td>
<td>$10</td>
<td>$27</td>
</tr>
<tr>
<td>$30</td>
<td>$40</td>
<td>$40</td>
<td>$25</td>
</tr>
<tr>
<td>$40</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$20</td>
<td>$30</td>
<td>$2</td>
<td></td>
</tr>
</tbody>
</table>

\[ V_S(A) = \frac{5}{10}(20) + \frac{3}{10}(30) + \frac{2}{10}(40) = 10 + 9 + 8 = 27 \]
\[ V_S(B) = \frac{5}{10}(10) + \frac{3}{10}(40) + \frac{2}{10}(40) = 5 + 12 + 8 = 25 \]

Alternative A has greater expected monetary value.
Example

Alice consults Bob who advises her that, under its current financial position, the company’s preferences are:

\[ \begin{align*}
  \$20 & \sim \left[ \frac{3}{5} : \$40 | \frac{2}{5} : \$10 \right] \\
  \$30 & \sim \left[ \frac{4}{5} : \$40 | \frac{1}{5} : \$10 \right]
\end{align*} \]

The company’s utility for money is:

\[
\begin{array}{c|ccc|c}
 & 0 & \frac{3}{5} & \frac{4}{5} & 1 \\
\hline
\$10 & \$20 & \$30 & \$40 \\
\end{array}
\]

The utility table:

\[
\begin{array}{ccc|c}
 & \frac{5}{10} & \frac{3}{10} & \frac{2}{10} \\
\hline
\downarrow & - & \uparrow & U \\
A & \frac{3}{5} & \frac{4}{5} & 1 \\
B & 0 & 1 & 1 \\
\end{array}
\]

\[ U(A) = \frac{5}{10} \left( \frac{3}{5} \right) + \frac{3}{10} \left( \frac{4}{5} \right) + \frac{2}{10} (1) \]
\[ = \frac{1}{50} \left( 15 + 12 + 10 \right) = \frac{74}{100} \]

\[ U(B) = \frac{5}{10} (0) + \frac{3}{10} (1) + \frac{2}{10} (1) \]
\[ = \frac{1}{50} (0 + 15 + 10) = \frac{50}{100} \]

Therefore, A will also have greater utility.

Suppose Bob cannot give precise assessments on values of $20 and $30, only bounds:

\[ \left[ \frac{3}{5} \$40 \right] \succ \$20 \succ \left[ \frac{1}{2} \$40 \right] \]
\[ \$40 \succ \$30 \succ \left[ \frac{4}{5} \$40 \right] \]

The utility for money is:

\[
\begin{array}{c|ccc|c}
 & 0 & \frac{1}{2} & \frac{3}{5} & \frac{4}{5} & 1 \\
\hline
\$10 & \$20 & \$30 & \$40 \\
\end{array}
\]

Lower bound for A:

\[
\begin{array}{ccc|c}
 & \frac{5}{10} & \frac{3}{10} & \frac{2}{10} \\
\hline
\downarrow & - & \uparrow & U \\
A & \frac{1}{2} & \frac{4}{5} & 1 \\
B & 0 & 1 & 1 \\
\end{array}
\]

\[ U(A) = \frac{5}{10} \left( \frac{1}{2} \right) + \frac{3}{10} \left( \frac{4}{5} \right) + \frac{2}{10} (1) \]
\[ = \frac{1}{100} \left( 5 + 6 + 10 \right) = \frac{69}{100} \]

Upper bound for A:

\[
\begin{array}{ccc|c}
 & \frac{5}{10} & \frac{3}{10} & \frac{2}{10} \\
\hline
\downarrow & - & \uparrow & U \\
A & \frac{3}{5} & 1 & 1 \\
B & 0 & 1 & 1 \\
\end{array}
\]

\[ U(A) = \frac{5}{10} \left( \frac{3}{5} \right) + \frac{3}{10} \left( \frac{1}{1} \right) + \frac{2}{10} (1) \]
\[ = \frac{1}{100} \left( 3 + 3 + 10 \right) = \frac{80}{100} \]
Bounds on A:

\[ U(A) > \frac{5}{10} \left( \frac{1}{2} \right) + \frac{3}{10} \left( \frac{4}{5} \right) + \frac{2}{10} (1) \]
\[ = \frac{1}{100} (25 + 24 + 20) \]
\[ = \frac{69}{100} \]

\[ U(A) < \frac{5}{10} \left( \frac{3}{5} \right) + \frac{3}{10} (1) + \frac{2}{10} (1) \]
\[ = \frac{1}{100} (30 + 30 + 20) \]
\[ = \frac{80}{100} \]

That is:
\[ \frac{69}{100} < U(A) < \frac{80}{100} \]

Conclusion:
A is guaranteed to be preferred to B \((U(B) = \frac{50}{100})\) regardless of the uncertainty over the precise preference for $20 and $30.

Summary

- Explored decision problems in greater depth:
  - actions that affect epistemic state (value of information-gathering actions)
  - dealing with uncertainty in preferences (sensitivity analysis)
- Updating beliefs (epistemic state) via Bayes’s theorem
- Value of information: cost of gathering more information versus increase in expected utility due to new information
- Sensitivity analysis:
  - decisions under imprecise preferences
  - how might preference uncertainty affect a decision?