Exercise 1. A Boolean formula in Conjunctive Normal Form (CNF) is a conjunction (AND) of disjunctions (OR) of literals (a Boolean variable or its negation). A HORN formula is a CNF formula where each clause contains at most one positive literal. For a CNF formula \( F \) and an assignment \( \tau : S \to \{0,1\} \) to a subset \( S \) of its variables, the formula \( F[\tau] \) is obtained from \( F \) by removing each clause that contains a literal that evaluates to 1 under \( S \), and removing all literals that evaluate to 0 from the remaining clauses.

**HORN-Backdoor Detection**

Input: A CNF formula \( F \) and an integer \( k \).
Parameter: \( k \)
Question: Is there a subset \( S \) of the variables of \( F \) with \( |S| \leq k \) such that for each assignment \( \tau : S \to \{0,1\} \), the formula \( F[\tau] \) is a HORN formula?

Example: \((\neg a \lor b \lor c) \land (b \lor \neg c \lor \neg d) \land (a \lor b \lor \neg e) \land (\neg b \lor c \lor \neg e)\) with \( k = 1 \) is a Yes-instance, certified by \( S = \{b\} \).

- Show that HORN-Backdoor Detection is FPT using the fact that Vertex Cover is FPT.

**Hint.**
- Show the following: if two distinct positive literals occur in a same clause, then a HORN-backdoor must contain at least one of the corresponding variables.
- Construct a parameterized reduction to Vertex Cover based on these pairwise conflicts.

Exercise 2. Show that Weighted Circuit Satisfiability \( \in XP \).

**Hint.**
- There are \( n^k \) assignments of weight \( k \), where \( n \) is the number of input gates.

Exercise 3. Recall that a \( k \)-coloring of a graph \( G = (V,E) \) is a function \( f : V \to \{1,2,\ldots,k\} \) assigning colors to \( V \) such that no two adjacent vertices receive the same color.

**Multicolor Clique**

Input: A graph \( G = (V,E) \), an integer \( k \), and a \( k \)-coloring of \( G \)
Parameter: \( k \)
Question: Does \( G \) have a clique of size \( k \)?

- Show that Multicolor Clique is W[1]-hard.

**Hint:** Reduce from Clique, and create \( k \) copies of \( V \), each one being an independent set in \( G' \). Add edges to enforce constraints that a clique of size \( k \) in \( G' \) corresponds to a clique of size \( k \) in \( G \), and vice-versa.

**Solution.** The proof is by a parameterized reduction from Clique.

**Construction.** Let \((G = (V,E),k)\) be an instance for Clique. We construct an instance \((G' = (V',E'),k',f)\) for Multicolor Clique as follows. For each \( v \in V \), create \( k \) vertices \( v(1),\ldots,v(k) \) and add them to \( V' \). For every
pair \( u(i), v(j) \in V' \) with \( i \neq j \), add \( u(i)v(j) \) to \( E' \) if and only if \( uv \in E \). Set \( k' := k \). Set \( f(v(i)) = i \) for each \( v \in V \) and \( i \in \{1, \ldots, k\} \).

**Equivalence.** \( G \) has a clique of size \( k \) if and only if \( G' \) has a clique of size \( k \).

\((\Rightarrow)\): Let \( S = \{s_1, \ldots, s_k\} \) be a clique in \( G \). Then \( S' = \{s_1(1), s_2(2), \ldots, s_k(k)\} \) is a clique in \( G' \) since \( s_is_j \in E \) implies \( s_i(s_j(j)) \in E' \) in our construction.

\((\Leftarrow)\): Let \( S' \) be a clique of size \( k \) in \( G' \). Since for each \( i \in \{1, \ldots, k\} \), \( \{v_i : v \in V\} \) is an independent set in \( G' \), \( S' \) contains exactly one vertex from each color class of \( f \). Denote \( S' = \{s'_1(1), \ldots, s'_k(k)\} \). Then, \( S = \{s_1, \ldots, s_k\} \) is a clique in \( G \).

**Parameter.** \( k' \leq k \).

**Running time.** The construction can clearly be done in FPT time, and even in polynomial time.

**Exercise 4.** A set system \( S \) is a pair \((V, H)\), where \( V \) is a finite set of elements and \( H \) is a set of subsets of \( V \). A set cover of a set system \( S = (V, H) \) is a subset \( X \) of \( H \) such that each element of \( V \) is contained in at least one of the sets in \( X \), i.e., \( \bigcup_{Y \in X} Y = V \).

**Set Cover**

- **Input:** A set system \( S = (V, H) \) and an integer \( k \)
- **Parameter:** \( k \)
- **Question:** Does \( S \) have a set cover of cardinality at most \( k \)?

- Show that Set Cover is W[2]-hard.

**Hint.** Reduce from Dominating Set:
- add an element for each vertex and
- add a set for each vertex, containing all the vertices in its closed neighborhood.

**Exercise 5.** A hitting set of a set system \( S = (V, H) \) is a subset \( X \) of \( V \) such that \( X \) contains at least one element of each set in \( H \), i.e., \( X \cap Y \neq \emptyset \) for each \( Y \in H \).

**Hitting Set**

- **Input:** A set system \( S = (V, H) \) and an integer \( k \)
- **Parameter:** \( k \)
- **Question:** Does \( S \) have a hitting set of size at most \( k \)?

- Show that Hitting Set is W[2]-hard.

**Hint:** Exploit a duality between sets and elements in set covers and hitting sets.

**Solution sketch.** Reduce from Set Cover. Let \((S = (V, H), k)\) be an instance for Set Cover. Construct an instance \((S' = (V', H'), k)\) for Hitting Set:
- \( V' := H \)
- \( H' := \{\{h \in H : v \in h\} : v \in V\} \)