COMP2111 Week 3
Term 1, 2019
Propositional Logic III

Summary of topics

- Well-formed formulas
- Boolean Algebras
- Valuations
- CNF/DNF
- Proof
- Natural deduction

Motivation

Given a theory T and a formula φ, how do we show $T \models \varphi$?

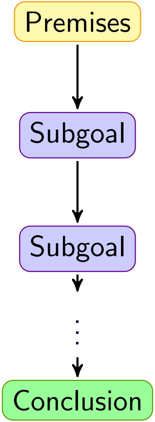
- Consider all valuations v (SEMANTIC approach)
- Use a sequence of **deductive rules** to show that φ is a logical consequence of T (SYNTACTIC approach)

Formal proofs

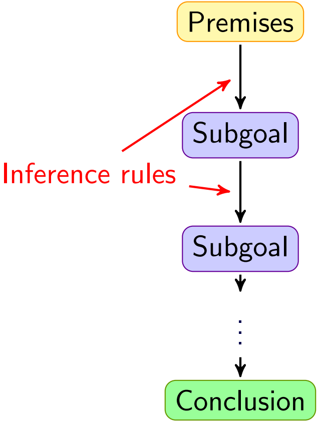
A formal way to show that a formula logically follows from a theory.

- Highly disciplined way of reasoning (good for computers)
- A sequence of formulas where each step is a deduction based on earlier steps
- Based entirely on rewriting formulas – no semantic interpretations needed

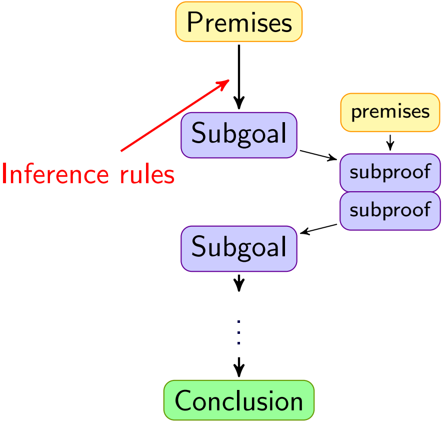
Proof stucture



Proof stucture



Proof stucture



Inference rules

In its simplest form, an **inference rule** is a statement of the form:

If I have a proof of this then I have a proof of that

Inference rules

In its simplest form, an **inference rule** is a statement of the form:

If I have a proof of this then I have a proof of that

A more complicated form:
If I have a proof of this (under these assumptions)
then I have a proof of that (under these* assumptions)

NB

The sets of assumptions need not be the same!

Inference rules

In its simplest form, an **inference rule** is a statement of the form:

If I have a proof of this then I have a proof of that

Yet more complicated form:

If I have a proof of this (under these assumptions)
and I have a proof of this (under these assumptions)
and ...
then I have a proof of that (under these assumptions)

NB

The sets of assumptions need not be the same!

Inference rules: notation

If T is a theory and φ is a formula, we write $T \vdash \varphi$ to denote a proof of φ under the assumptions T.

So an inference rule is a statement of the form:

If $T_1 \vdash \varphi_1$ and $T_2 \vdash \varphi_2$ and ... and $T_n \vdash \varphi_n$ then $T \vdash \psi$

Alternative notation:

$$\frac{T_1 \vdash \varphi_1 \qquad T_2 \vdash \varphi_2 \qquad \cdots \qquad T_n \vdash \varphi_n}{T \vdash \psi}$$

Inference rules: notation

If T is a theory and φ is a formula, we write $T \vdash \varphi$ to denote a proof of φ under the assumptions T.

So an inference rule is a statement of the form:

If $T_1 \vdash \varphi_1$ and $T_2 \vdash \varphi_2$ and ... and $T_n \vdash \varphi_n$ then $T \vdash \psi$

Alternative notation:

$$\frac{T_1 \vdash \varphi_1 \qquad T_2 \vdash \varphi_2 \qquad \cdots \qquad T_n \vdash \varphi_n}{T \vdash \psi}$$

Inference rules: notation

If T is a theory and φ is a formula, we write $T \vdash \varphi$ to denote a proof of φ under the assumptions T.

So an inference rule is a statement of the form:

If $T_1 \vdash \varphi_1$ and $T_2 \vdash \varphi_2$ and ... and $T_n \vdash \varphi_n$ then $T \vdash \psi$

Alternative notation:

$$\frac{\varphi_1 \qquad \varphi_2 \qquad \cdots \qquad \varphi_n}{\psi}$$

(If $T_1 = T_2 = \ldots = T$)

Inference rules: notation

If T is a theory and φ is a formula, we write $T \vdash \varphi$ to denote a proof of φ under the assumptions T.

So an inference rule is a statement of the form:

If $T_1 \vdash \varphi_1$ and $T_2 \vdash \varphi_2$ and ... and $T_n \vdash \varphi_n$ then $T \vdash \psi$

Alternative notation:

$$\frac{\varphi_1 \qquad \begin{array}{c}[\alpha] \\ \vdots \\ \varphi_2\end{array} \qquad \cdots \qquad \varphi_n}{\psi}$$

(If $T_2 = T \cup \{\alpha\}$)

Inference rules: examples

$\wedge$-elimination:

$$\frac{A \wedge B}{A}\ (\wedge\text{-E1})$$

$\wedge$-introduction:

$$\frac{A \qquad B}{A \wedge B}\ (\wedge\text{-I})$$

$\vee$-elimination:

$$\frac{A \vee B \qquad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \qquad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}\ (\vee\text{-E})$$

Inference rules: examples

$\wedge$-elimination:

$$\frac{A \wedge B}{A}\ (\wedge\text{-E1})$$

$\wedge$-introduction:

$$\frac{A \qquad B}{A \wedge B}\ (\wedge\text{-I})$$

$\vee$-elimination:

$$\frac{A \vee B \qquad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C}\ (\vee\text{-E})$$

Inference rules: examples

$\wedge$-elimination:

$$\frac{A \wedge B}{A}\ (\wedge\text{-E1})$$

$\wedge$-introduction:

$$\frac{A \qquad B}{A \wedge B}\ (\wedge\text{-I})$$

$\vee$-elimination:

$$\frac{A \vee B \qquad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \qquad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}\ (\vee\text{-E})$$

Proof layout

Default layout: tabular

Example:

$$\text{Prove:} \quad A \to B, B \to C \vdash A \to C$$

Line	Premises	Formula	Rule	References
1		$A \to B$	Premise	
2		$B \to C$	Premise	
3		A	Premise	
4	1, 3	B	$\to$ -E	1, 3
5	1, 2, 3	C	$\to$ -E	2, 4
6	1, 2	$A \to C$	$\to$ -I	5

Proof layout

Default layout: tabular

Example:

$$\text{Prove:} \quad A \to B, B \to C \vdash A \to C$$

Line	Premises	Formula	Rule	References
1		$A \to B$	Premise	
2		$B \to C$	Premise	
3		A	Premise	
4	1, 3	B	$\to$ -E	1, 3
5	1, 2, 3	C	$\to$ -E	2, 4
6	1, 2	$A \to C$	$\to$ -I	5

Proof layout

Default layout: tabular
Example:

$$\text{Prove:} \quad A \to B, B \to C \vdash A \to C$$

Line	Premises	Formula	Rule	References
1		$A \to B$	Premise	
2		$B \to C$	Premise	
3		A	Premise	
4	$1, 3$	B	$\to$ -E	$1, 3$
5	$1, 2, 3$	C	$\to$ -E	$2, 4$
6	$1, 2$	$A \to C$	$\to$ -I	5

Proof layout

Default layout: tabular

Example:

$$\text{Prove:} \quad A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$$

Line	Premises	Formula	Rule	References
1		$A \rightarrow B$	Premise	
2		$B \rightarrow C$	Premise	
3		A	Premise	
4	$1, 3$	B	$\rightarrow$ -E	$1, 3$
5	$1, 2, 3$	C	$\rightarrow$ -E	$2, 4$
6	$1, 2$	$A \rightarrow C$	$\rightarrow$ -I	5

Proof layout

Default layout: tabular

Example:

Prove: $A \to B, B \to C \vdash A \to C$

Line	Premises	Formula	Rule	References
1		$A \to B$	Premise	
2		$B \to C$	Premise	
3		A	Premise	
4	$1, 3$	B	$\to$-E	$1, 3$
5	$1, 2, 3$	C	$\to$-E	$2, 4$
6	$1, 2$	$A \to C$	$\to$-I	5

Proof layout

Default layout: tabular

Example:

$$\text{Prove:} \quad A \to B, B \to C \vdash A \to C$$

Line	**Premises**	**Formula**	**Rule**	**References**
1		$A \to B$	Premise	
2		$B \to C$	Premise	
3		A	Premise	
4	$1, 3$	B	$\to$ -E	$1, 3$
5	$1, 2, 3$	C	$\to$ -E	$2, 4$
6	$1, 2$	$A \to C$	$\to$ -I	5

Proof layout

Default layout: tabular
Example:

$$\text{Prove:} \quad A \to B, B \to C \vdash A \to C$$

Line	Premises	Formula	Rule	References
1		$A \to B$	Premise	
2		$B \to C$	Premise	
3		A	Premise	
4	$1, 3$	B	$\to$ -E	$1, 3$
5	$1, 2, 3$	C	$\to$ -E	$2, 4$
6	$1, 2$	$A \to C$	$\to$ -I	5

Premises

You may assume anything, but it may not be helpful!
You must **discharge** any assumptions you make along the way.

For example,

Prove: $A \to B, B \to C \vdash B \wedge C$

Line	Premises	Formula	Rule	References
1		$A \to B$	Premise	
2		$B \to C$	Premise	
3		A	Premise	
4	$1, 3$	B	$\to$ -E	$1, 3$
5	$1, 2, 3$	C	$\to$ -E	$2, 4$
6	$1, 2, 3$	$B \wedge C$	$\wedge$-I	$4, 5$

Premises

You may assume anything, but it may not be helpful!
You must **discharge** any assumptions you make along the way.

For example,

$$\text{Prove:} \quad A \to B, B \to C \vdash B \wedge C$$

Line	Premises	Formula	Rule	References
1		$A \to B$	Premise	
2		$B \to C$	Premise	
3		A	Premise	
4	$1, 3$	B	$\to$ -E	$1, 3$
5	$1, 2, 3$	C	$\to$ -E	$2, 4$
6	$1, 2, 3$	$B \wedge C$	$\wedge$-I	$4, 5$

Other proof layouts: Proof tree

Upside-down tree (root at bottom):

- Natural structure arising from rule syntax
- Premises at leaves
- Conclusion at root

$$\dfrac{\dfrac{B \to C \qquad \dfrac{A \to B \qquad [A]}{B}\,(\to\text{-E})}{C}\,(\to\text{-E})}{A \to C}\,(\to\text{-I})$$

Other proof layouts: Fitch-style

- Style used in online tool
- Premises ruled off from subgoals
- Subproofs are indented

1. $A \rightarrow B$	
2. $B \rightarrow C$	
3. A	
4. B	$\rightarrow$-E: 1,3
5. C	$\rightarrow$-E: 2,4
6. $A \rightarrow C$	$\rightarrow$-I: 3–5

Proof systems

There are many proof systems, defined by the default **axioms** ("free" premises), and **inference rules**:

- Natural deduction
- Hilbert systems
- Sequent calculus
- Resolution

Proof systems

Two key properties of a "good" system:

Soundness: The system only proves valid statements: If $T \vdash \varphi$ then $T \models \varphi$.

Completeness: The system can prove any valid statement: If $T \models \varphi$ then $T \vdash \varphi$.

NB

Soundness is straightforward: check every axiom and inference rule.

Completeness (with consistency) is trickier, and may not even exist (e.g. Gödel's incompleteness theorem).

Summary of topics

- Well-formed formulas
- Boolean Algebras
- Valuations
- CNF/DNF
- Proof
- Natural deduction

Online resources

Open logic project (https://proofs.openlogicproject.org/)

- Freely available textbooks
- Online proof checker
- Sample exercises

Natural deduction

Proof system intended to mirror "natural reasoning":

- No axioms (no inherent truths)
- 15 inference rules: primarily grouped into pairs (and trios) of rules tasked with **introducing** and **eliminating** boolean operators from the chain of reasoning.

Operator	Introduction	Elimination
$\wedge$	$\wedge$-I	$\wedge$-E1 $\quad \wedge$-E2
$\vee$	$\vee$-I1 $\quad \vee$-I2	$\vee$-E
$\rightarrow$	$\rightarrow$-I	$\rightarrow$-E
$\leftrightarrow$	$\leftrightarrow$-I	$\leftrightarrow$-E1 $\quad \leftrightarrow$-E2
$\neg$	$\neg$-I	$\neg$-E $\quad$ IP
$\perp$	$\neg$-E	X

$\wedge$ Introduction and Elimination

$\wedge$-introduction: $$\frac{A \qquad B}{A \wedge B}\ (\wedge\text{-I})$$

$\wedge$-elimination (1): $$\frac{A \wedge B}{A}\ (\wedge\text{-E1})$$

$\wedge$-elimination (2): $$\frac{A \wedge B}{B}\ (\wedge\text{-E2})$$

$\wedge$ Introduction and Elimination

$\wedge$-introduction:

$$\frac{A \qquad B}{A \wedge B}\ (\wedge\text{-I})$$

$\wedge$-elimination (1):

$$\frac{A \wedge B}{A}\ (\wedge\text{-E1})$$

$\wedge$-elimination (2):

$$\frac{A \wedge B}{B}\ (\wedge\text{-E2})$$

Proof example

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

Line	Premises	Formula	Rule	References
1		$(A \wedge B) \wedge C$	Premise	
2	1	$A \wedge B$	$\wedge$-E1	1
3	1	A	$\wedge$-E1	2
4	1	B	$\wedge$-E2	2
5	1	C	$\wedge$-E2	1
6	1	$B \wedge C$	$\wedge$-I	4, 5
7	1	$A \wedge (B \wedge C)$	$\wedge$-I	3, 6

Proof example

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

Line	Premises	Formula	Rule	References
1		$(A \wedge B) \wedge C$	Premise	
2	1	$A \wedge B$	$\wedge$-E1	1
3	1	A	$\wedge$-E1	2
4	1	B	$\wedge$-E2	2
5	1	C	$\wedge$-E2	1
6	1	$B \wedge C$	$\wedge$-I	4, 5
7	1	$A \wedge (B \wedge C)$	$\wedge$-I	3, 6

Proof example

Prove: $(A \land B) \land C \vdash A \land (B \land C)$

Line	Premises	Formula	Rule	References
1		$(A \land B) \land C$	Premise	
2	1	$A \land B$	$\land$-E1	1
3	1	A	$\land$-E1	2
4	1	B	$\land$-E2	2
5	1	C	$\land$-E2	1
6	1	$B \land C$	$\land$-I	$4, 5$
7	1	$A \land (B \land C)$	$\land$-I	$3, 6$

Proof example

Prove: $(A \land B) \land C \vdash A \land (B \land C)$

Line	Premises	Formula	Rule	References
1		$(A \land B) \land C$	Premise	
2	1	$A \land B$	$\land$-E1	1
3	1	A	$\land$-E1	2
4	1	B	$\land$-E2	2
5	1	C	$\land$-E2	1
6	1	$B \land C$	$\land$-I	4, 5
7	1	$A \land (B \land C)$	$\land$-I	3, 6

Proof example

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

Line	Premises	Formula	Rule	References
1		$(A \wedge B) \wedge C$	Premise	
2	1	$A \wedge B$	$\wedge$-E1	1
3	1	A	$\wedge$-E1	2
4	1	B	$\wedge$-E2	2
5	1	C	$\wedge$-E2	1
6	1	$B \wedge C$	$\wedge$-I	$4, 5$
7	1	$A \wedge (B \wedge C)$	$\wedge$-I	$3, 6$

Proof example

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

Line	Premises	Formula	Rule	References
1		$(A \wedge B) \wedge C$	Premise	
2	1	$A \wedge B$	$\wedge$-E1	1
3	1	A	$\wedge$-E1	2
4	1	B	$\wedge$-E2	2
5	1	C	$\wedge$-E2	1
6	1	$B \wedge C$	$\wedge$-I	4, 5
7	1	$A \wedge (B \wedge C)$	$\wedge$-I	3, 6

Proof example

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

Line	Premises	Formula	Rule	References
1		$(A \wedge B) \wedge C$	Premise	
2	1	$A \wedge B$	$\wedge$-E1	1
3	1	A	$\wedge$-E1	2
4	1	B	$\wedge$-E2	2
5	1	C	$\wedge$-E2	1
6	1	$B \wedge C$	$\wedge$-I	4, 5
7	1	$A \wedge (B \wedge C)$	$\wedge$-I	3, 6

Proof example

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

Line	Premises	Formula	Rule	References
1		$(A \wedge B) \wedge C$	Premise	
2	1	$A \wedge B$	$\wedge$-E1	1
3	1	A	$\wedge$-E1	2
4	1	B	$\wedge$-E2	2
5	1	C	$\wedge$-E2	1
6	1	$B \wedge C$	$\wedge$-I	$4, 5$
7	1	$A \wedge (B \wedge C)$	$\wedge$-I	$3, 6$

Proof example (Fitch)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

1. $(A \wedge B) \wedge C$	
2. $A \wedge B$	$\wedge$-E1: 1
3. A	$\wedge$-E1: 2
4. B	$\wedge$-E2: 2
5. C	$\wedge$-E2: 1
6. $B \wedge C$	$\wedge$-I: 4,5
7. $A \wedge (B \wedge C)$	$\wedge$-I: 3,6

Proof example (Fitch)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

1. $(A \wedge B) \wedge C$	
2. $A \wedge B$	$\wedge$-E1: 1
3. A	$\wedge$-E1: 2
4. B	$\wedge$-E2: 2
5. C	$\wedge$-E2: 1
6. $B \wedge C$	$\wedge$-I: 4,5
7. $A \wedge (B \wedge C)$	$\wedge$-I: 3,6

Proof example (Fitch)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

1. $(A \wedge B) \wedge C$	
2. $A \wedge B$	$\wedge$-E1: 1
3. A	$\wedge$-E1: 2
4. B	$\wedge$-E2: 2
5. C	$\wedge$-E2: 1
6. $B \wedge C$	$\wedge$-I: 4,5
7. $A \wedge (B \wedge C)$	$\wedge$-I: 3,6

Proof example (Fitch)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

1. (A ∧ B) ∧ C	
2. A ∧ B	∧-E1: 1
3. A	∧-E1: 2
4. B	∧-E2: 2
5. C	∧-E2: 1
6. B ∧ C	∧-I: 4,5
7. A ∧ (B ∧ C)	∧-I: 3,6

Proof example (Fitch)

Prove: $(A \land B) \land C \vdash A \land (B \land C)$

1. $(A \land B) \land C$	
2. $A \land B$	$\land$-E1: 1
3. A	$\land$-E1: 2
4. B	$\land$-E2: 2
5. C	$\land$-E2: 1
6. $B \land C$	$\land$-I: 4,5
7. $A \land (B \land C)$	$\land$-I: 3,6

Proof example (Fitch)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

1. $(A \wedge B) \wedge C$	
2. $A \wedge B$	$\wedge$-E1: 1
3. A	$\wedge$-E1: 2
4. B	$\wedge$-E2: 2
5. C	$\wedge$-E2: 1
6. $B \wedge C$	$\wedge$-I: 4,5
7. $A \wedge (B \wedge C)$	$\wedge$-I: 3,6

Proof example (Fitch)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

1. (A ∧ B) ∧ C	
2. A ∧ B	∧-E1: 1
3. A	∧-E1: 2
4. B	∧-E2: 2
5. C	∧-E2: 1
6. B ∧ C	∧-I: 4,5
7. A ∧ (B ∧ C)	∧-I: 3,6

Proof example (Tree)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

$(A \wedge B) \wedge C$

(∧-E1)

$A \wedge B$

(∧-E2)

B

(∧-I)

$(A \wedge B) \wedge C$

(∧-E1)

$A \wedge B$

(∧-E1)

A

(∧-I)

$(A \wedge B) \wedge C$

(∧-E2)

C

$B \wedge C$

$A \wedge (B \wedge C)$

Proof example (Tree)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

$$(A \wedge B) \wedge C$$

$$\dfrac{(A \wedge B) \wedge C}{A \wedge B}\ (\wedge\text{-E1})$$

$$(A \wedge B) \wedge C$$

Proof example (Tree)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

$$\dfrac{\dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\,(\wedge\text{-E1})}{A}\,(\wedge\text{-E1}) \qquad \dfrac{\dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\,(\wedge\text{-E1})}{B}\,(\wedge\text{-E2}) \qquad \dfrac{(A \wedge B) \wedge C}{C}\,(\wedge\text{-E2})}{B \wedge C}\,(\wedge\text{-I})}{A \wedge (B \wedge C)}\,(\wedge\text{-I})$$

Proof example (Tree)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

$$\dfrac{\dfrac{\dfrac{(A\wedge B)\wedge C}{A\wedge B}\,(\wedge\text{-E1})}{A}\,(\wedge\text{-E1}) \qquad \dfrac{\dfrac{\dfrac{(A\wedge B)\wedge C}{A\wedge B}\,(\wedge\text{-E1})}{B}\,(\wedge\text{-E2}) \qquad \dfrac{(A\wedge B)\wedge C}{C}\,(\wedge\text{-E2})}{B\wedge C}\,(\wedge\text{-I})}{A\wedge(B\wedge C)}\,(\wedge\text{-I})$$

Proof example (Tree)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

$$\dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\;(\wedge\text{-E1})}{A}\;(\wedge\text{-E1}) \qquad \dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\;(\wedge\text{-E1})}{B}\;(\wedge\text{-E2}) \qquad (A \wedge B) \wedge C$$

$(\wedge\text{-E2})$ C

$(\wedge\text{-I})$ $B \wedge C$

$(\wedge\text{-I})$ $A \wedge (B \wedge C)$

Proof example (Tree)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

$$\dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\,(\wedge\text{-E1})}{A}\,(\wedge\text{-E1}) \qquad \dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\,(\wedge\text{-E1})}{B}\,(\wedge\text{-E2}) \qquad \dfrac{(A \wedge B) \wedge C}{C}\,(\wedge\text{-E2})$$

$(\wedge\text{-I})$ $(\wedge\text{-I})$ $B \wedge C$

$A \wedge (B \wedge C)$

Proof example (Tree)

Prove: $(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$

$$
\dfrac{
\dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\,(\wedge\text{-E1})}{A}\,(\wedge\text{-E1})
\qquad
\dfrac{
\dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}\,(\wedge\text{-E1})}{B}\,(\wedge\text{-E2})
\qquad
\dfrac{(A \wedge B) \wedge C}{C}\,(\wedge\text{-E2})
}{B \wedge C}\,(\wedge\text{-I})
}{A \wedge (B \wedge C)}\,(\wedge\text{-I})
$$

Proof example (Tree)

Prove: $(A \land B) \land C \vdash A \land (B \land C)$

$$
\dfrac{
  \dfrac{
    \dfrac{(A \land B) \land C}{A \land B}\,(\land\text{-E1})
  }{A}\,(\land\text{-E1})
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{(A \land B) \land C}{A \land B}\,(\land\text{-E1})
    }{B}\,(\land\text{-E2})
    \qquad
    \dfrac{(A \land B) \land C}{C}\,(\land\text{-E2})
  }{B \land C}\,(\land\text{-I})
}{A \land (B \land C)}\,(\land\text{-I})
$$

$\vee$ Introduction and Elimination

$\vee$-introduction (1): $$\frac{A}{A \vee B}\ (\vee\text{-I1})$$

$\vee$-introduction (2): $$\frac{B}{A \vee B}\ (\vee\text{-I1})$$

$\vee$-elimination: $$\frac{A \vee B \quad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}\ (\vee\text{-E})$$

$\vee$ Introduction and Elimination

$\vee$-introduction (1): $$\frac{A}{A \vee B}\ (\vee\text{-I1})$$

$\vee$-introduction (2): $$\frac{B}{A \vee B}\ (\vee\text{-I1})$$

$\vee$-elimination: $$\frac{A \vee B \qquad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \qquad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}\ (\vee\text{-E})$$

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	Premises	Formula	Rule	References
1		$A \vee (B \wedge C)$	Premise	
2		A	Premise	
3	2	$A \vee B$	$\vee$-I1	2
4	2	$A \vee C$	$\vee$-I1	2
5	2	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	3, 4
6		$(B \wedge C)$	Premise	
7	6	B	$\wedge$-E1	6
8	6	$A \vee B$	$\vee$-I2	7
9	6	C	$\wedge$-E2	6
10	6	$A \vee C$	$\vee$-I2	9
11	6	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	8, 10
12	1	$(A \vee B) \wedge (A \vee C)$	$\vee$-E	5, 11

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	Premises	Formula	Rule	References
1		$A \vee (B \wedge C)$	Premise	
2		A	Premise	
3	2	$A \vee B$	$\vee$-I1	2
4	2	$A \vee C$	$\vee$-I1	2
5	2	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	3, 4
6		$(B \wedge C)$	Premise	
7	6	B	$\wedge$-E1	6
8	6	$A \vee B$	$\vee$-I2	7
9	6	C	$\wedge$-E2	6
10	6	$A \vee C$	$\vee$-I2	9
11	6	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	8, 10
12	1	$(A \vee B) \wedge (A \vee C)$	$\vee$-E	5, 11

Proof example

Prove: $A \lor (B \land C) \vdash (A \lor B) \land (A \lor C)$

Line	Premises	Formula	Rule	References
1		$A \lor (B \land C)$	Premise	
2		A	Premise	
3	2	$A \lor B$	$\lor$-I1	2
4	2	$A \lor C$	$\lor$-I1	2
5	2	$(A \lor B) \land (A \lor C)$	$\land$-I	3, 4
6		$(B \land C)$	Premise	
7	6	B	$\land$-E1	6
8	6	$A \lor B$	$\lor$-I2	7
9	6	C	$\land$-E2	6
10	6	$A \lor C$	$\lor$-I2	9
11	6	$(A \lor B) \land (A \lor C)$	$\land$-I	8, 10
12	1	$(A \lor B) \land (A \lor C)$	$\lor$-E	5, 11

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	Premises	Formula	Rule	References
1		$A \vee (B \wedge C)$	Premise	
2		A	Premise	
3	2	$A \vee B$	$\vee$-I1	2
4	2	$A \vee C$	$\vee$-I1	2
5	2	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	3, 4
6		$(B \wedge C)$	Premise	
7	6	B	$\wedge$-E1	6
8	6	$A \vee B$	$\vee$-I2	7
9	6	C	$\wedge$-E2	6
10	6	$A \vee C$	$\vee$-I2	9
11	6	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	8, 10
12	1	$(A \vee B) \wedge (A \vee C)$	$\vee$-E	5, 11

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	Premises	Formula	Rule	References
1		$A \vee (B \wedge C)$	Premise	
2		A	Premise	
3	2	$A \vee B$	$\vee$-I1	2
4	2	$A \vee C$	$\vee$-I1	2
5	2	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	3, 4
6		$(B \wedge C)$	Premise	
7	6	B	$\wedge$-E1	6
8	6	$A \vee B$	$\vee$-I2	7
9	6	C	$\wedge$-E2	6
10	6	$A \vee C$	$\vee$-I2	9
11	6	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	8, 10
12	1	$(A \vee B) \wedge (A \vee C)$	$\vee$-E	5, 11

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	Premises	Formula	Rule	References
1		$A \vee (B \wedge C)$	Premise	
2		A	Premise	
3	2	$A \vee B$	$\vee$-I1	2
4	2	$A \vee C$	$\vee$-I1	2
5	2	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	3, 4
6		$(B \wedge C)$	Premise	
7	6	B	$\wedge$-E1	6
8	6	$A \vee B$	$\vee$-I2	7
9	6	C	$\wedge$-E2	6
10	6	$A \vee C$	$\vee$-I2	9
11	6	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	8, 10
12	1	$(A \vee B) \wedge (A \vee C)$	$\vee$-E	5, 11

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	Premises	Formula	Rule	References
1		$A \vee (B \wedge C)$	Premise	
2		A	Premise	
3	2	$A \vee B$	$\vee$-I1	2
4	2	$A \vee C$	$\vee$-I1	2
5	2	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	3, 4
6		$(B \wedge C)$	Premise	
7	6	B	$\wedge$-E1	6
8	6	$A \vee B$	$\vee$-I2	7
9	6	C	$\wedge$-E2	6
10	6	$A \vee C$	$\vee$-I2	9
11	6	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	8, 10
12	1	$(A \vee B) \wedge (A \vee C)$	$\vee$-E	5, 11

Proof example

Prove: $A \lor (B \land C) \vdash (A \lor B) \land (A \lor C)$

Line	Premises	Formula	Rule	References
1		$A \lor (B \land C)$	Premise	
2		A	Premise	
3	2	$A \lor B$	$\lor$-I1	2
4	2	$A \lor C$	$\lor$-I1	2
5	2	$(A \lor B) \land (A \lor C)$	$\land$-I	3, 4
6		$(B \land C)$	Premise	
7	6	B	$\land$-E1	6
8	6	$A \lor B$	$\lor$-I2	7
9	6	C	$\land$-E2	6
10	6	$A \lor C$	$\lor$-I2	9
11	6	$(A \lor B) \land (A \lor C)$	$\land$-I	8, 10
12	1	$(A \lor B) \land (A \lor C)$	$\lor$-E	5, 11

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	Premises	Formula	Rule	References
1		$A \vee (B \wedge C)$	Premise	
2		A	Premise	
3	2	$A \vee B$	$\vee$-I1	2
4	2	$A \vee C$	$\vee$-I1	2
5	2	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	3, 4
6		$(B \wedge C)$	Premise	
7	6	B	$\wedge$-E1	6
8	6	$A \vee B$	$\vee$-I2	7
9	6	C	$\wedge$-E2	6
10	6	$A \vee C$	$\vee$-I2	9
11	6	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	8, 10
12	1	$(A \vee B) \wedge (A \vee C)$	$\vee$-E	5, 11

Proof example

Prove: $A \lor (B \land C) \vdash (A \lor B) \land (A \lor C)$

Line	Premises	Formula	Rule	References
1		$A \lor (B \land C)$	Premise	
2		A	Premise	
3	2	$A \lor B$	$\lor$-I1	2
4	2	$A \lor C$	$\lor$-I1	2
5	2	$(A \lor B) \land (A \lor C)$	$\land$-I	3, 4
6		$(B \land C)$	Premise	
7	6	B	$\land$-E1	6
8	6	$A \lor B$	$\lor$-I2	7
9	6	C	$\land$-E2	6
10	6	$A \lor C$	$\lor$-I2	9
11	6	$(A \lor B) \land (A \lor C)$	$\land$-I	8, 10
12	1	$(A \lor B) \land (A \lor C)$	$\lor$-E	5, 11

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	Premises	Formula	Rule	References
1		$A \vee (B \wedge C)$	Premise	
2		A	Premise	
3	2	$A \vee B$	$\vee$-I1	2
4	2	$A \vee C$	$\vee$-I1	2
5	2	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	3, 4
6		$(B \wedge C)$	Premise	
7	6	B	$\wedge$-E1	6
8	6	$A \vee B$	$\vee$-I2	7
9	6	C	$\wedge$-E2	6
10	6	$A \vee C$	$\vee$-I2	9
11	6	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	8, 10
12	1	$(A \vee B) \wedge (A \vee C)$	$\vee$-E	5, 11

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	Premises	Formula	Rule	References
1		$A \vee (B \wedge C)$	Premise	
2		A	Premise	
3	2	$A \vee B$	$\vee$-I1	2
4	2	$A \vee C$	$\vee$-I1	2
5	2	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	3, 4
6		$(B \wedge C)$	Premise	
7	6	B	$\wedge$-E1	6
8	6	$A \vee B$	$\vee$-I2	7
9	6	C	$\wedge$-E2	6
10	6	$A \vee C$	$\vee$-I2	9
11	6	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	8, 10
12	1	$(A \vee B) \wedge (A \vee C)$	$\vee$-E	5, 11

Proof example

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

Line	Premises	Formula	Rule	References
1		$A \vee (B \wedge C)$	Premise	
2		A	Premise	
3	2	$A \vee B$	$\vee$-I1	2
4	2	$A \vee C$	$\vee$-I1	2
5	2	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	3, 4
6		$(B \wedge C)$	Premise	
7	6	B	$\wedge$-E1	6
8	6	$A \vee B$	$\vee$-I2	7
9	6	C	$\wedge$-E2	6
10	6	$A \vee C$	$\vee$-I2	9
11	6	$(A \vee B) \wedge (A \vee C)$	$\wedge$-I	8, 10
12	1	$(A \vee B) \wedge (A \vee C)$	$\vee$-E	5, 11

Proof example (Fitch)

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

1. $A \vee (B \wedge C)$	
2. A	
3. $A \vee B$	$\vee$-I1: 2
4. $A \vee C$	$\vee$-I1: 2
5. $(A \vee B) \wedge (A \vee C)$	$\wedge$-I: 3,4
6. $B \wedge C$	
7. B	$\wedge$-E1: 6
8. $A \vee B$	$\vee$-I2: 7
9. C	$\wedge$-E2: 6
10. $A \vee C$	$\vee$-I2: 9
11. $(A \vee B) \wedge (A \vee C)$	$\wedge$-I: 8,10
12. $(A \vee B) \wedge (A \vee C)$	$\vee$-E: 1,3–5,6–11

Proof example (Fitch)

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

1. $A \vee (B \wedge C)$	
2. A	
3. $A \vee B$	$\vee$-I1: 2
4. $A \vee C$	$\vee$-I1: 2
5. $(A \vee B) \wedge (A \vee C)$	$\wedge$-I: 3,4
6. $B \wedge C$	
7. B	$\wedge$-E1: 6
8. $A \vee B$	$\vee$-I2: 7
9. C	$\wedge$-E2: 6
10. $A \vee C$	$\vee$-I2: 9
11. $(A \vee B) \wedge (A \vee C)$	$\wedge$-I: 8,10
12. $(A \vee B) \wedge (A \vee C)$	$\vee$-E: 1,3–5,6–11

Proof example (Fitch)

Prove: $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

1. $A \vee (B \wedge C)$	
2. A	
3. $A \vee B$	$\vee$-I1: 2
4. $A \vee C$	$\vee$-I1: 2
5. $(A \vee B) \wedge (A \vee C)$	$\wedge$-I: 3,4
6. $B \wedge C$	
7. B	$\wedge$-E1: 6
8. $A \vee B$	$\vee$-I2: 7
9. C	$\wedge$-E2: 6
10. $A \vee C$	$\vee$-I2: 9
11. $(A \vee B) \wedge (A \vee C)$	$\wedge$-I: 8,10
12. $(A \vee B) \wedge (A \vee C)$	$\vee$-E: 1,3–5,6–11

$\rightarrow$ Introduction and Elimination

$\rightarrow$-introduction:

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \rightarrow B}\;(\rightarrow\text{-I})$$

$\rightarrow$-elimination:
(Modus Ponens)

$$\dfrac{A \rightarrow B \qquad A}{B}\;(\rightarrow\text{-E})$$

$\leftrightarrow$ Introduction and Elimination

$\leftrightarrow$-introduction:

$$\dfrac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix} \qquad \begin{matrix}[B] \\ \vdots \\ A\end{matrix}}{A \leftrightarrow B}\ (\leftrightarrow\text{-I})$$

$\leftrightarrow$-elimination (1):

$$\dfrac{A \leftrightarrow B \qquad A}{B}\ (\leftrightarrow\text{-E1})$$

$\leftrightarrow$-elimination (2):

$$\dfrac{A \leftrightarrow B \qquad B}{A}\ (\leftrightarrow\text{-E1})$$

$\neg$ Introduction and Elimination and Indirect Proof

$\neg$-introduction:

$$\dfrac{\begin{array}{c}[A]\\ \vdots\\ \bot\end{array}}{\neg A}\;(\neg\text{-I})$$

$\neg$-elimination:
($\bot$-introduction)

$$\dfrac{A \qquad \neg A}{\bot}\;(\neg\text{-E})$$

Indirect proof:

$$\dfrac{\begin{array}{c}[\neg A]\\ \vdots\\ \bot\end{array}}{A}\;(\text{IP})$$

$\neg$ Introduction and Elimination and Indirect Proof

$\neg$-introduction:

$$\dfrac{\begin{array}{c}[A]\\ \vdots\\ \bot\end{array}}{\neg A}\ (\neg\text{-I})$$

$\neg$-elimination:
($\bot$-introduction)

$$\dfrac{A \qquad \neg A}{\bot}\ (\neg\text{-E})$$

Indirect proof:

$$\dfrac{\begin{array}{c}[\neg A]\\ \vdots\\ \bot\end{array}}{A}\ (\text{IP})$$

Proof example: double negation

Prove: $A \vdash \neg\neg A$

Line	Premises	Formula	Rule	References
1		A	Premise	
2		$\neg A$	Premise	
3	$1, 2$	$\perp$	$\neg$-E	$1, 2$
4	1	$\neg\neg A$	$\neg$-I	3

Proof example: double negation

Prove: $A \vdash \neg\neg A$

Line	Premises	Formula	Rule	References
1		A	Premise	
2		$\neg A$	Premise	
3	1, 2	$\perp$	$\neg$-E	1, 2
4	1	$\neg\neg A$	$\neg$-I	3

Proof example: double negation

Prove: $A \vdash \neg\neg A$

Line	Premises	Formula	Rule	References
1		A	Premise	
2		$\neg A$	Premise	
3	1, 2	$\bot$	$\neg$-E	1, 2
4	1	$\neg\neg A$	$\neg$-I	3

Proof example: double negation

Prove: $A \vdash \neg\neg A$

Line	Premises	Formula	Rule	References
1		A	Premise	
2		$\neg A$	Premise	
3	$1, 2$	$\bot$	$\neg$-E	$1, 2$
4	1	$\neg\neg A$	$\neg$-I	3

Proof example: double negation

Prove: $\neg\neg A \vdash A$

Line	Premises	Formula	Rule	References
1		$\neg\neg A$	Premise	
2		$\neg A$	Premise	
3	1, 2	$\bot$	$\neg$-E	1, 2
4	1	A	IP	3

Proof example: double negation

Prove: $\neg\neg A \vdash A$

Line	Premises	Formula	Rule	References
1		$\neg\neg A$	Premise	
2		$\neg A$	Premise	
3	1, 2	$\bot$	$\neg$-E	1, 2
4	1	A	IP	3

Proof example: double negation

Prove: $\neg\neg A \vdash A$

Line	Premises	Formula	Rule	References
1		$\neg\neg A$	Premise	
2		$\neg A$	Premise	
3	$1, 2$	$\bot$	$\neg$-E	$1, 2$
4	1	A	IP	3

Proof example: double negation

Prove: $\neg\neg A \vdash A$

Line	Premises	Formula	Rule	References
1		$\neg\neg A$	Premise	
2		$\neg A$	Premise	
3	1, 2	$\perp$	$\neg$-E	1, 2
4	1	A	?IP	3

Proof example: double negation

Prove: $\neg\neg A \vdash A$

Line	Premises	Formula	Rule	References
1		$\neg\neg A$	Premise	
2		$\neg A$	Premise	
3	$1, 2$	$\perp$	$\neg$-E	$1, 2$
4	1	A	IP	3

Explosion

Explosion:
($\perp$-elimination)

$$\frac{\perp}{A}\;(\text{X})$$

Soundness and completeness

Theorem

Natural deduction is sound and complete. That is,

$$T \vdash \varphi \quad \textit{if and only if} \quad T \models \varphi$$

Corollary

The following are equivalent:

- $\varphi_1, \varphi_2, \ldots, \varphi_n \models \varphi$
- $\varphi_1 \wedge \cdots \wedge \varphi_n) \to \varphi$ *is a tautology*
- $\varphi_1 \to (\varphi_2 \to (\cdots \to (\varphi_n \to \varphi)) \cdots)$ *is a tautology*
- $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \varphi$
- $\varphi_1, \varphi_2, \ldots, \varphi_{n-1} \vdash \varphi_n \to \varphi$
- *(and so on)*

Soundness and completeness

Theorem

Natural deduction is sound and complete. That is,

$$T \vdash \varphi \quad \textit{if and only if} \quad T \models \varphi$$

Corollary

The following are equivalent:

- $\varphi_1, \varphi_2, \ldots, \varphi_n \models \varphi$
- $\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow \varphi$ *is a tautology*
- $\varphi_1 \rightarrow (\varphi_2 \rightarrow (\cdots \rightarrow (\varphi_n \rightarrow \varphi))\cdots)$ *is a tautology*
- $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \varphi$
- $\varphi_1, \varphi_2, \ldots, \varphi_{n-1} \vdash \varphi_n \rightarrow \varphi$
- *(and so on)*

Derived rules

Several useful rules available in the proof checker (not needed for Assignment!)

Double negation elimination: $$\frac{\neg\neg A}{A}\ (\mathrm{DNE})$$

Reiteration: $$\frac{A}{A}\ (\mathrm{R})$$

Law of excluded middle: $$\frac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix} \qquad \begin{matrix}[\neg A] \\ \vdots \\ B\end{matrix}}{B}\ (\mathrm{LEM})$$

Derived rules

Disjunctive syllogism:

$$\frac{A \vee B \qquad \neg A}{B}\ (\mathrm{DS})$$

Modus Tollens:

$$\frac{A \rightarrow B \qquad \neg B}{\neg A}\ (\mathrm{MT})$$

De Morgans Laws (e.g.):

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}\ (\mathrm{DM})$$

Summary of topics

- Well-formed formulas
- Boolean Algebras
- Valuations
- CNF/DNF
- Proof
- Natural deduction