Tractable Reasoning with Limited Belief

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COMP4418, Week 8

Computational Aspects of Reasoning

- Good news:

  $\mathbf{O}\mathrm{KB} \models \alpha$ reduces to $\mathrm{KB} \models \phi_1, \ldots, \mathrm{KB} \models \phi_k$ (Representation Theorem)

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- Bad (old) news:
  First-order case: $\mathrm{KB} \models \phi$ is only semidecidable
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  Propositional case: $\mathrm{KB} \models \phi$ is intractable (or P = NP)

  - $\mathrm{KB} \models \phi$ is co-NP-complete
  - co-NP contains all problems whose complement is in NP

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2. We could restrict the expressivity of our representation language.
   - Horn logic
   - Description logics

   But: Humans can deal with very complex representations.

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- Knowledge is closed under equivalence
  E.g., $\models \mathbf{K}p \leftrightarrow \mathbf{K}(p \wedge (q \vee \neg q))$
- Inconsistent knowledge implies knowing everything (incl. nonsense)
  E.g., $\models \mathbf{K}(p \wedge \neg p) \rightarrow \mathbf{K}q$

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This is different from restricting expressiveness:

- Horn logic, description logics restrict the *language*
- Limited belief restricts the *semantics* (mainly)

Overview of the Lecture

- **Limited Belief – First Attempt**
- Limited Belief – Second Attempt
- Data structures and algorithms for ASP solvers

Limited Belief — First Attempt

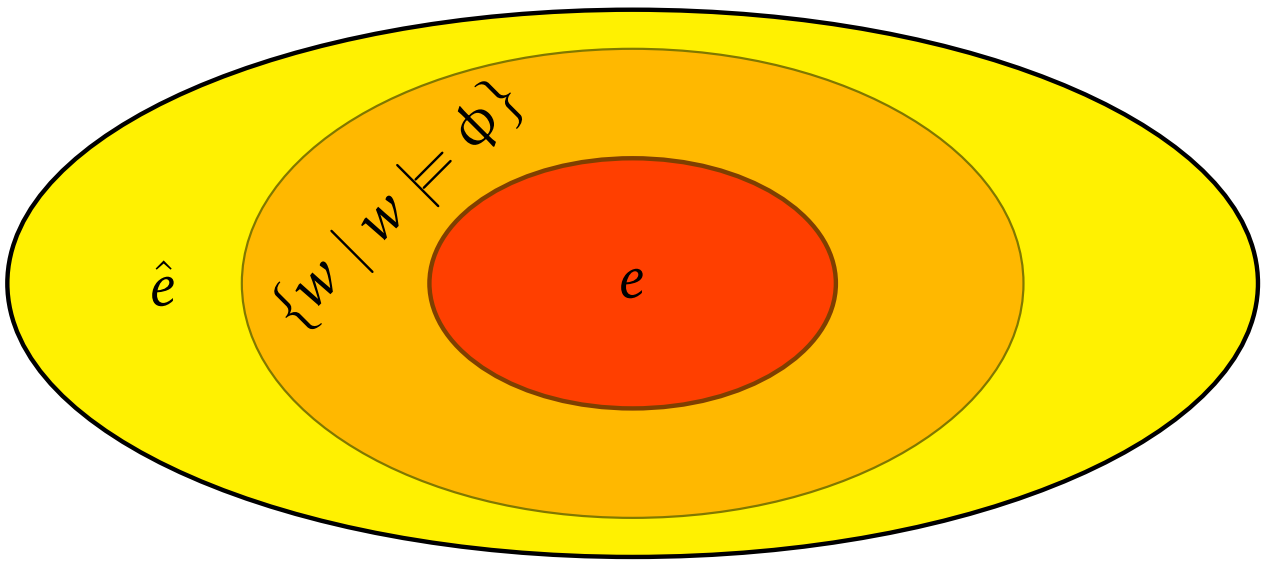
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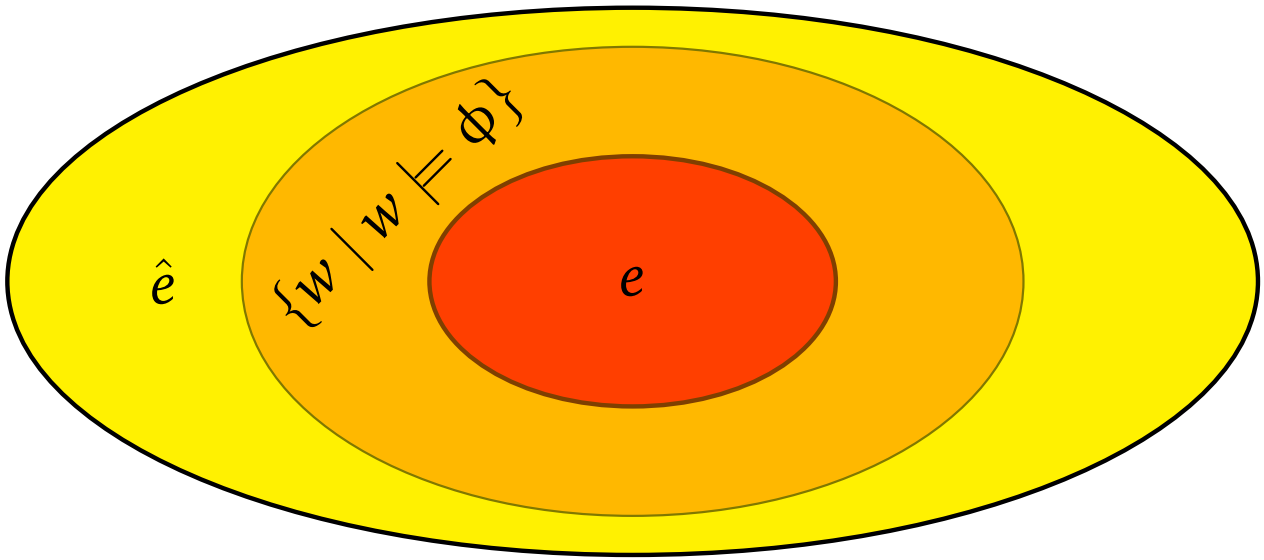


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Simplification: propositional logic for now, no nested $\mathbf{O}$, $\mathbf{K}$.

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- $v[p] = \{0\}$ means "false"
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We will define *true support* and *false support*.

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An **epistemic state** e is a set of multi-valued worlds.

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$\mathbf{O}\mathrm{KB} \mid\!\approx \mathbf{K}\alpha \iff$ for all e, v, $e, v \models_{\mathrm{T}} \mathbf{O}\mathrm{KB} \Rightarrow e, v \models_{\mathrm{T}} \mathbf{K}\alpha$

Examples

Let $e \models_{\mathrm{T}} \mathbf{O}((p \vee q \vee r) \wedge (p \vee q \vee \neg r))$.

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  $= \{v \mid v \models_{\mathrm{T}} p$ or $v \models_{\mathrm{T}} q$ or $v \models_{\mathrm{T}} r$, and
  $v \models_{\mathrm{T}} p$ or $v \models_{\mathrm{T}} q$ or $v \models_{\mathrm{F}} r\}$

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So $\mathbf{O}((p \vee q \vee r) \wedge (p \vee q \vee \neg r))$ really doesn't entail much...

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- Inconsistent knowledge does not imply knowing everything
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Is reasoning easier now?

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Let $p_1, \ldots, p_n$ be the propositions in KB and ϕ.

$$\mathrm{KB} \models \phi \iff \mathbf{O}(\mathrm{KB} \wedge \underbrace{\textstyle\bigwedge_i (p_i \vee \neg p_i)}_{\text{prevent “conflicting information”}}) \mid\!\approx \mathbf{K}(\phi \vee \underbrace{\textstyle\bigvee_i (p_i \wedge \neg p_i)}_{\text{ignore “never heard of” worlds}})$$

Complexity (2)

Good news: Reasoning gets very easy when KB and ϕ are in CNF.

Theorem: decision procedure for CNF KB, ϕ

Let $\mathrm{KB} \stackrel{\text{def}}{=} c_1 \wedge \ldots \wedge c_m$ and $\phi \stackrel{\text{def}}{=} d_1 \wedge \ldots \wedge d_n$ for clauses c_i, d_j.

$\mathbf{O}\mathrm{KB} \mid\!\approx \mathbf{K}\phi$ is decidable in $\mathcal{O}(m \cdot n)$.

$\mathbf{O}\mathrm{KB} \mid\!\approx \mathbf{K}\phi \iff$ for every d_j, there is a c_i with $c_i \subseteq d_j$.

Ex.: $\mathbf{O}((p \vee \neg q) \wedge q) \mid\!\approx \mathbf{K}(p \vee \neg q \vee r)$ since $\{p, \neg q\} \subseteq \{p, \neg q, r\}$.

$\mathbf{O}((p \vee \neg q) \wedge q) \not\mid\!\approx \mathbf{K}p$ since $\{p, \neg q\} \not\subseteq \{p\}$, $\{q\} \not\subseteq \{p\}$.

Proof on paper.

The First-Order Case

Generalise to first-order $\mathcal{OL}$ (function symbols aside):

- Predicates: $P(t_1, \ldots, t_j)$ where t_i is variable or standard name
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Generalise the true and false support semantics to this language:

Definition: multi-valued world, first-order case

$P(\vec{n})$ is **primitive** iff all n_i are standard names.
A **multi-valued world** v is a function from the primitive atomic formulas to $\{\{\}, \{0\}, \{1\}, \{0, 1\}\}$.

- $e, v \models_{\mathrm{T}} \exists x\, \alpha \iff e, v \models_{\mathrm{T}} \alpha^x_n$ for some standard name n
  $e, v \models_{\mathrm{F}} \exists x\, \alpha \iff e, v \models_{\mathrm{F}} \alpha^x_n$ for every standard name n

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Bad news: Too complex.

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Overview of the Lecture

- Limited Belief – First Attempt
- **Limited Belief – Second Attempt**
- Data structures and algorithms for ASP solvers

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Simplification: propositional logic for now, no nested $\mathbf{O}$, $\mathbf{K}$.

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- If $P(n) \in \mathrm{KB}$ and $\forall x\,(P(x) \to Q(x)) \in \mathrm{KB}$, then $\mathbf{O}\mathrm{KB} \mid\!\approx \mathbf{K}_0\,Q(n)$

Hard Inferences

What should not count as explicit belief?

Things that are not obvious (requires to consider different cases).

For example, only one of the following KBs entails $\exists x\,(P(x) \wedge Q(x))$:

$\mathbf{KB_1}$	$\mathbf{KB_2}$
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Let $w \models P(a) \wedge P(d) \wedge Q(b) \wedge Q(c)$.
Then $w \models \mathrm{KB}_1$ but $w \not\models \exists x\,(P(x) \wedge Q(x))$.

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Let $w \models \mathrm{KB}_2$.

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Let $w \models \mathrm{KB}_2 \wedge P(a)$.

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Let $w \models \mathrm{KB}_2 \wedge P(a) \wedge \neg Q(a) \wedge P(b) \wedge \neg Q(b) \wedge P(g) \wedge Q(g)$.
Then $w \models \exists x\,(P(x) \wedge Q(x))$.

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Semantic representation: set of clauses instead of set of worlds

- Set of worlds $\approx$ disjunction of conjunctions (DNF)
- Set of clauses $\approx$ conjunction of disjunctions (CNF)
- CNF is often more compact than DNF

Setups, Unit Propagation, Subsumption

- Identify clause $\ell_1 \vee \ldots \vee \ell_j$ with $\{\ell_1, \ldots, \ell_j\}$
- We write $\overline{\ell}$ to flip the sign of ℓ, e.g., $\overline{p}$ is $\neg p$, and $\overline{\neg p}$ is p
- Recall: empty clause is unsatisfiable

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Definition: unit propagation, subsumption, setup

A **setup** s is a (possibly infinite) set of ground clauses.
Unit propagation infers $c \setminus \{\overline{\ell}\}$ from c and ℓ.
Subsumption infers $c \cup d$ from c.
$\mathrm{UP}(s)$ closes s under unit propagation.
$\mathrm{UP}^+(s)$ adds subsumed clauses.
$\mathrm{UP}^-(s)$ removes subsumed clauses.

Examples

Ex.: $c_1 = (p \vee q \vee r)$, $c_2 = (p \vee q \vee \neg r)$

- $\mathrm{UP}(\{c_1, c_2\}) = \{c_1, c_2\}$
- $\mathrm{UP}(\{c_1, c_2, r\}) = \{c_1, c_2, r, (p \vee q)\}$
- $\mathrm{UP}(\{c_1, c_2, \neg r\}) = \{c_1, c_2, \neg r, (p \vee q)\}$
- $\mathrm{UP}^+(\{c_1, c_2, \neg r\}) = \{c_1, c_2, \neg r, (p \vee q)\} \cup \{c \mid c \supseteq \neg r \text{ or } c \supseteq (p \vee q)\}$
- $\mathrm{UP}^-(\{c_1, c_2, \neg r\}) = \{\neg r, (p \vee q)\}$

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Unit propagation = forward chaining

$\mathrm{UP}(s)$ can be computed in linear time (if s is finite).

Semantics of Limited Belief

Definition: semantics of limited belief

- $s \models\!\!\approx c \iff c \in \mathsf{UP}^+(s)$ if c is a clause
- $s \models\!\!\approx (\alpha \vee \beta) \iff s \models\!\!\approx \alpha$ or $s \models\!\!\approx \beta$ if $(\alpha \vee \beta)$ is not a clause

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- $s \mid\!\approx \mathbf{K}_0 \phi \iff s$ is *obviously inconsistent* or $s \mid\!\approx \phi$
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Let $s \mid\!\approx \mathbf{O}((p \vee q \vee r) \wedge (p \vee q \vee \neg r))$.

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  because UP infers $(p \vee q)$ from $(p \vee q \vee \neg r)$ and r

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Need a dual operator to $\mathbf{K}_k \phi$, call it $\mathbf{M}_k \phi$, to say that ϕ is possible.

Semantics of Limited Belief (2)

The semantics of unlimited $\mathbf{M}\,\alpha$ in $\mathcal{OL}$ is:

Definition: semantics $\mathbf{M}$

- $e, w \models \mathbf{M}\,\alpha \iff$ for some w, $w \in e$ and $e, w \models \alpha$

Note: $e, w \models \mathbf{M}\,\alpha \iff e, w \models \neg\mathbf{K}\neg\alpha$

Definition: semantics $\mathbf{M}_k$

- $s \mid\!\approx \mathbf{M}_0\,\phi \iff s$ is *obviously consistent* and $s \mid\!\approx \phi$
- $s \mid\!\approx \mathbf{M}_{k+1}\,\phi \iff$ for some atom P, $s \cup \{P\} \mid\!\approx \mathbf{M}_k\,\phi$
- $s \mid\!\approx \neg\mathbf{M}_k\,\phi \iff s \not\mid\!\approx \mathbf{M}_k\,\phi$

s is *obviously consistent* if no atomic proposition p occurs positively and negatively in clauses in $\mathrm{UP}^{-}(s)$.

Examples

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  - $\iff$ s is obv. consistent and $s \mid\!\approx p$
  - $\iff$ s is obv. consistent and $p \in \mathrm{UP}^+(s)$ ✗
    because s is not obv. consistent (r occurs pos. and neg. in $\mathrm{UP}^-(s)$)
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Let $s \mid\!\approx \mathbf{O}((p \vee q \vee r) \wedge (p \vee q \vee \neg r))$.

- $\mathsf{UP}^+(s) = \mathsf{UP}^+(\{(p \vee q \vee r),\ (p \vee q \vee \neg r)\})$
- $s \mid\!\approx \mathbf{K}_0(p \vee q)$ ✗
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- $s \mid\!\approx \mathbf{M}_1 p$
  - $\Longleftarrow$ $s \cup \{p\} \mid\!\approx \mathbf{M}_0 p$
  - $\Longleftrightarrow$ $s \cup \{p\}$ is obv. consistent and $s \cup \{p\} \mid\!\approx p$ ✓
    because s is obv. consistent ($\mathsf{UP}^-(s \cup \{p\}) = \{p\}$)
    and $p \in \mathsf{UP}^+(s \cup \{p\})$

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- $s \mid\!\approx \mathbf{M}_2 \neg p$ ✓

Some Properties

Theorem: monotonicity

$\models\!\!\approx \mathbf{K}_k \phi \rightarrow \mathbf{K}_{k+1} \phi.$
$\models\!\!\approx \mathbf{M}_k \phi \rightarrow \mathbf{M}_{k+1} \phi.$

Definition: proper$^+$ KB

A KB is proper$^+$ when it is a conjunction of clauses (CNF).

Let KB be proper$^+$ of the form $c_1 \wedge \ldots \wedge c_j$.

Theorem: unique-model property

$s \models\!\!\approx \mathbf{O}\mathrm{KB} \iff \mathrm{UP}^+(s) = \mathrm{UP}^+(\{c_1, \ldots, c_j\}).$

Some Properties (2)

Let KB be proper$^+$.

Theorem: soundness

$\mathbf{O}\mathrm{KB} \mid\!\approx \mathbf{K}_k \phi \implies \mathbf{O}\mathrm{KB} \models \mathbf{K}\phi.$
$\mathbf{O}\mathrm{KB} \mid\!\approx \mathbf{M}_k \phi \implies \mathbf{O}\mathrm{KB} \models \mathbf{M}\phi.$

Theorem: eventual completeness

$\mathbf{O}\mathrm{KB} \models \mathbf{K}\phi \implies \mathbf{O}\mathrm{KB} \mid\!\approx \mathbf{K}_k \phi$ for large enough k.
$\mathbf{O}\mathrm{KB} \models \mathbf{M}\phi \implies \mathbf{O}\mathrm{KB} \mid\!\approx \mathbf{M}_k \phi$ for large enough k.

Theorem: complexity

$\mathbf{O}\mathrm{KB} \models \mathbf{K}\phi$ and $\mathbf{O}\mathrm{KB} \models \mathbf{M}\phi$ is tractable for small k:

$$\mathcal{O}(2^k \cdot (|\mathrm{KB}| + |\phi|)^{k+3}).$$

Generalisation of the Logic of Limited Belief

- Introspection
  - Extend semantics to keep track of original setup without splits
  - Representation theorem translates to limited belief
- First-order logic
  - $s \mid\!\approx \exists x\, \phi \iff s \mid\!\approx \phi^x_n$ for some n
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Theorem: decidability

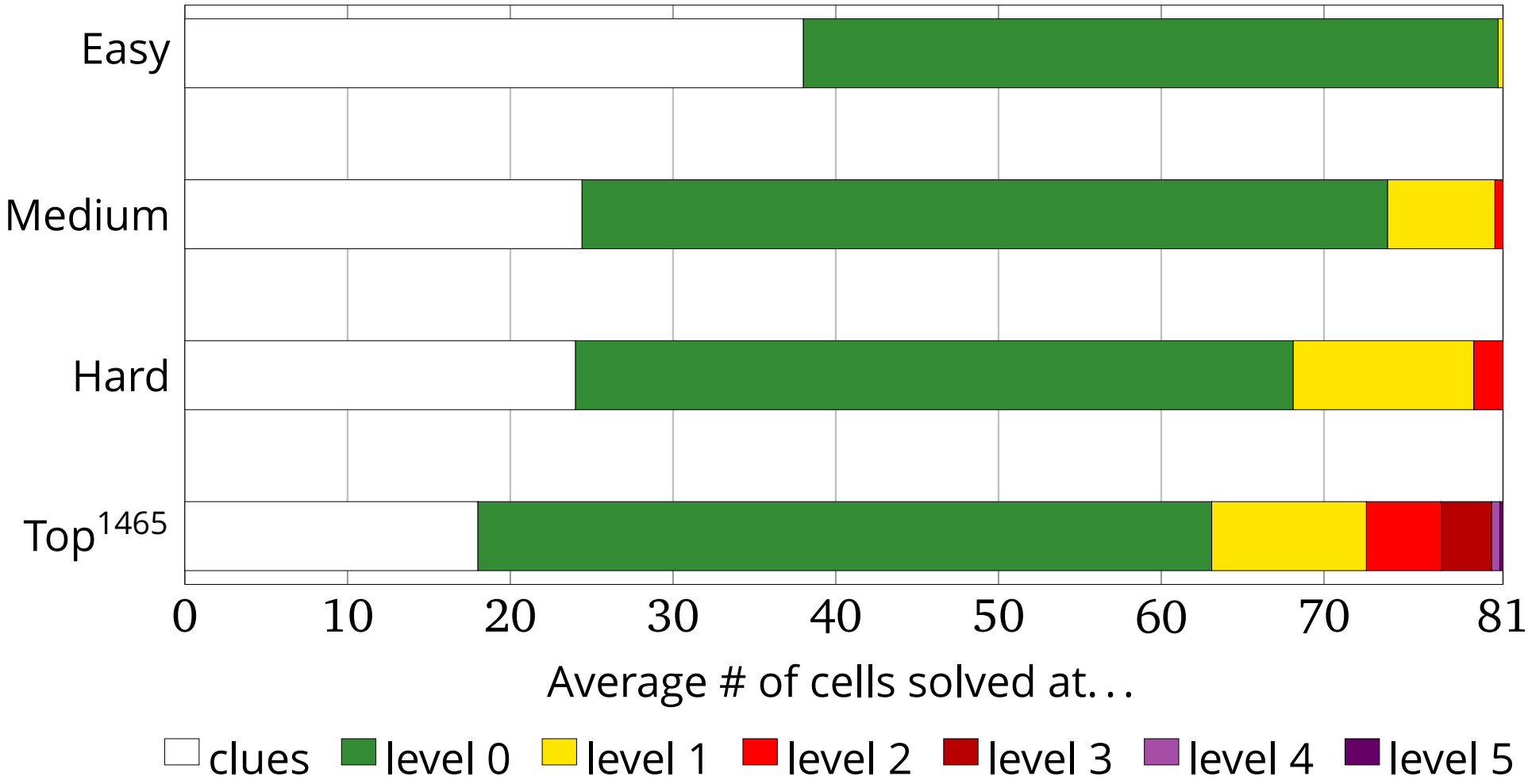
$\mathbf{O}\mathrm{KB} \mid\!\approx \sigma$ is decidable.

Does Limited Belief Work?

Experiment: Sudoku

- Fill 9×9 board with numbers $1,\ldots,9$ such that no identical numbers in rows, columns, 3×3 blocks
- Has a unique solution
- Difficulty depends on how many and which clues we get
  - Newspaper: easy ($\approx$ 38 clues), medium ($\approx$ 24 clues), hard ($\approx$ 24 clues)
  - Top 1465: extremely difficult (18 clues, proven minimum is 17)
- Question: do belief level and difficulty correlate?

Sudoku with Limited Belief



Overview of the Lecture

- Limited Belief – First Attempt
- Limited Belief – Second Attempt
- **Implementation Techniques**

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- Same techniques can be used for
  - Propositional Satisfiability
  - Answer Set Programs
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- Data structures and algorithms:
  - Davis-Putnam-Logemann-Loveland (DPLL) algorithm
  - Watched-Literal Scheme
  - Conflict-Driven Clause Learning (CDCL)

While SAT is NP-complete for propositional logic and for ASP, modern solvers can solve large instances (millions of variables).

DPLL Algorithm

Definition: DPLL algorithm

A literal ℓ is **assigned** in s iff $\ell \in s$ or $\overline{\ell} \in s$.

Input: set of clauses s
Output: 1 iff s is satisfiable in propositional logic

DPLL(s) procedure:

1. If s contains the empty clause, return 0
2. If all literals are assigned in s, return 1
3. Select some unassigned literal ℓ
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How to select literal? Prefer ones that trigger UP

Watched-Literal Scheme

- DPLL uses backtracking:
  1. Add ℓ to s, close under unit propagation
  2. (recursive calls)
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Watched-Literal Scheme: Observation

Let s be a set of clauses which is closed under unit propagation. Let $c \in s$ with $|c| \geq 2$. If c contains at least two unassigned literals, select two of them as *watched* literals; otherwise select two of them randomly. When we add a new literal ℓ to s, then unit propagation of c with all the unit clauses in s together with ℓ produces a new unit clause *only if* $\overline{\ell}$ is one of the watched literals.

Why?

- Case 1: Suppose there are two unassigned literals in c that are not assigned initially. Then the watched literals ℓ_1, ℓ_2 are unassigned. Suppose unit propagation of c with all the unit clauses in s together with ℓ produces a new unit clause c'. Then $|c'| = 1 < |c|$, so either $\ell_1 \notin c$ or $\ell_2 \notin c$. Since ℓ_1, ℓ_2 were not assigned before adding ℓ, either ℓ_1 or ℓ_2 must be $\overline{\ell}$.
- Case 2: There are no two unassigned literals in c. Then there is at most one unassigned literal ℓ_1 in c, in which case we have ℓ_1 already as a unit clause in s.

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AddUnit(ℓ) procedure:

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2. If $\overline{\ell} \in s$, mark s as inconsistent and return.
3. For every $c \in s$ with $|c| \geq 2$, check if $\overline{\ell}$ is watched.
   If yes, propagate the unit clauses from s with c to infer c'.
   If $|c'| = 0$, mark s as inconsistent.
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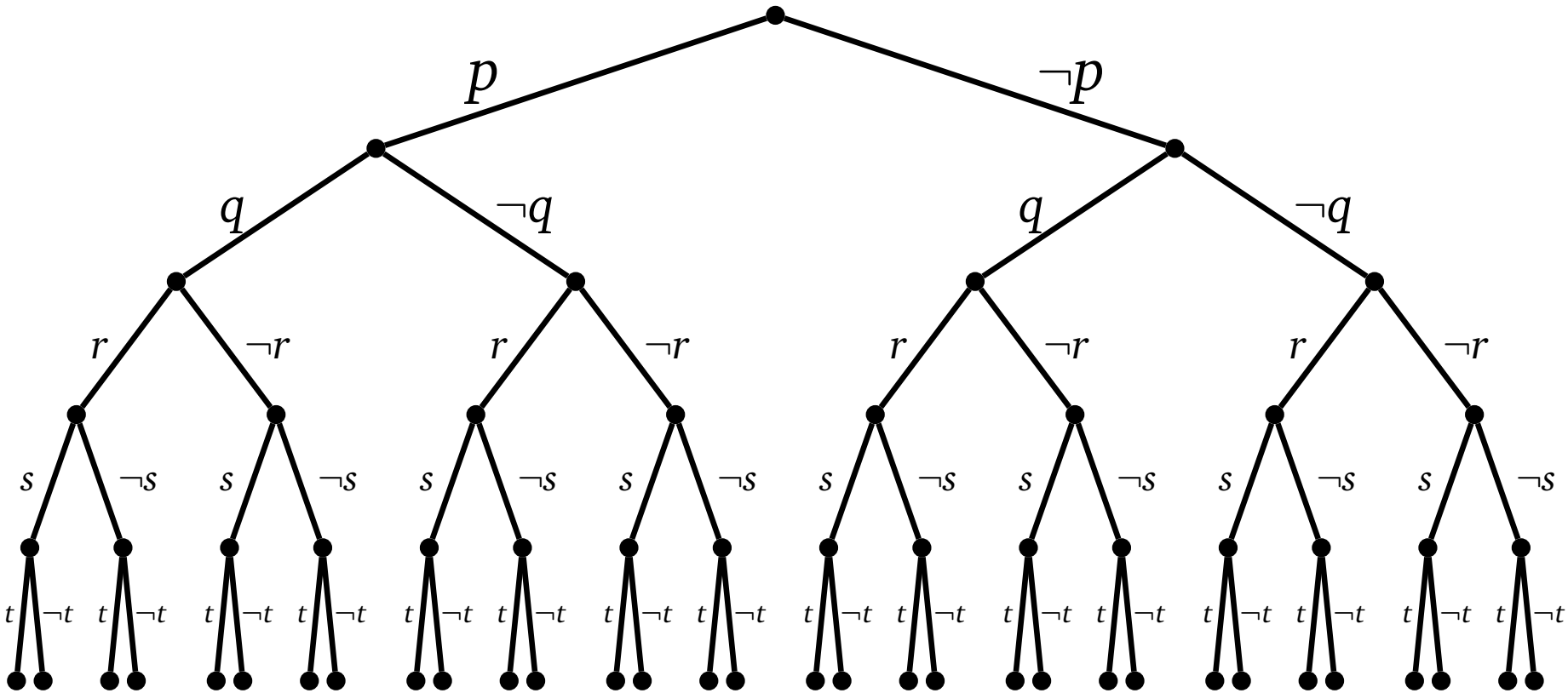
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Backtrack procedure:

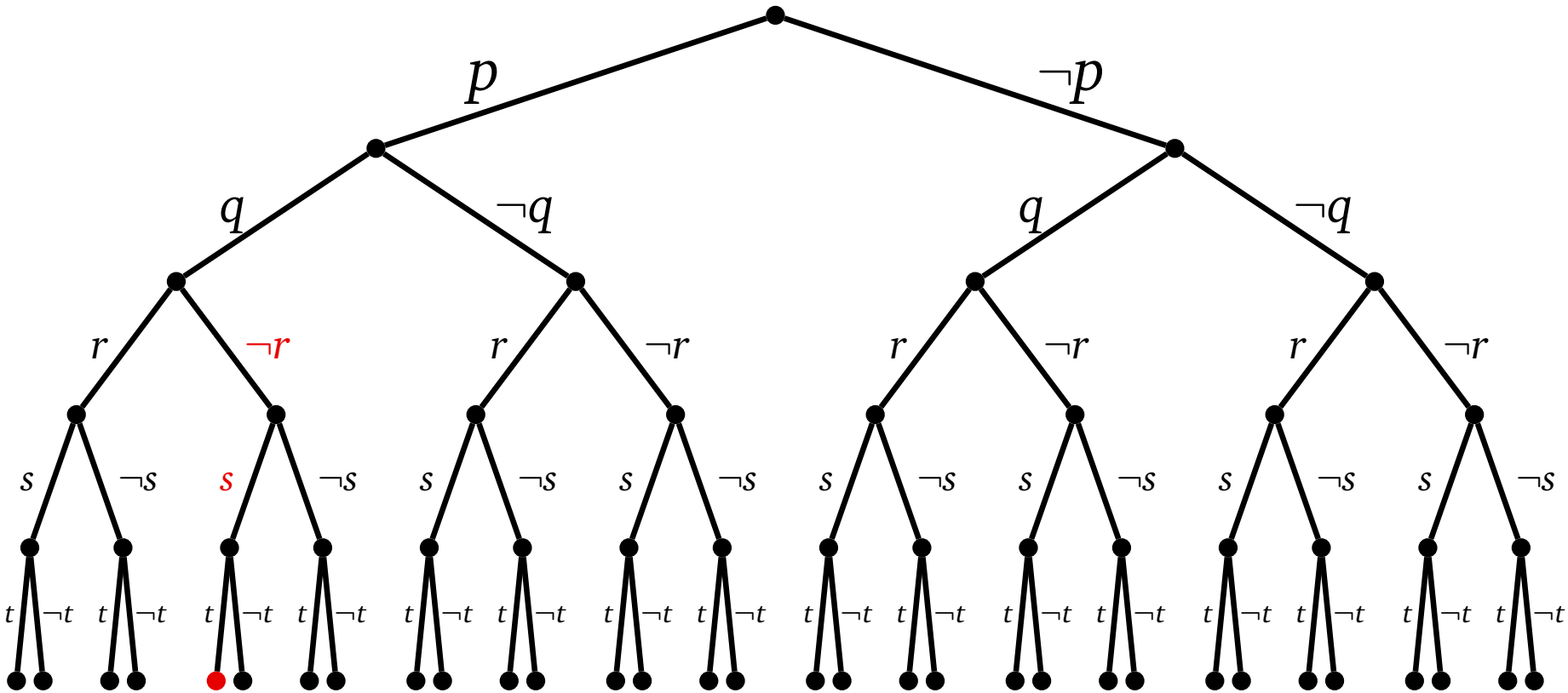
1. Store $n := |s|$
2. (recursive calls)
3. Pop from s until $|s| = n$

Example on paper

Conflict-Driven Clause Learning

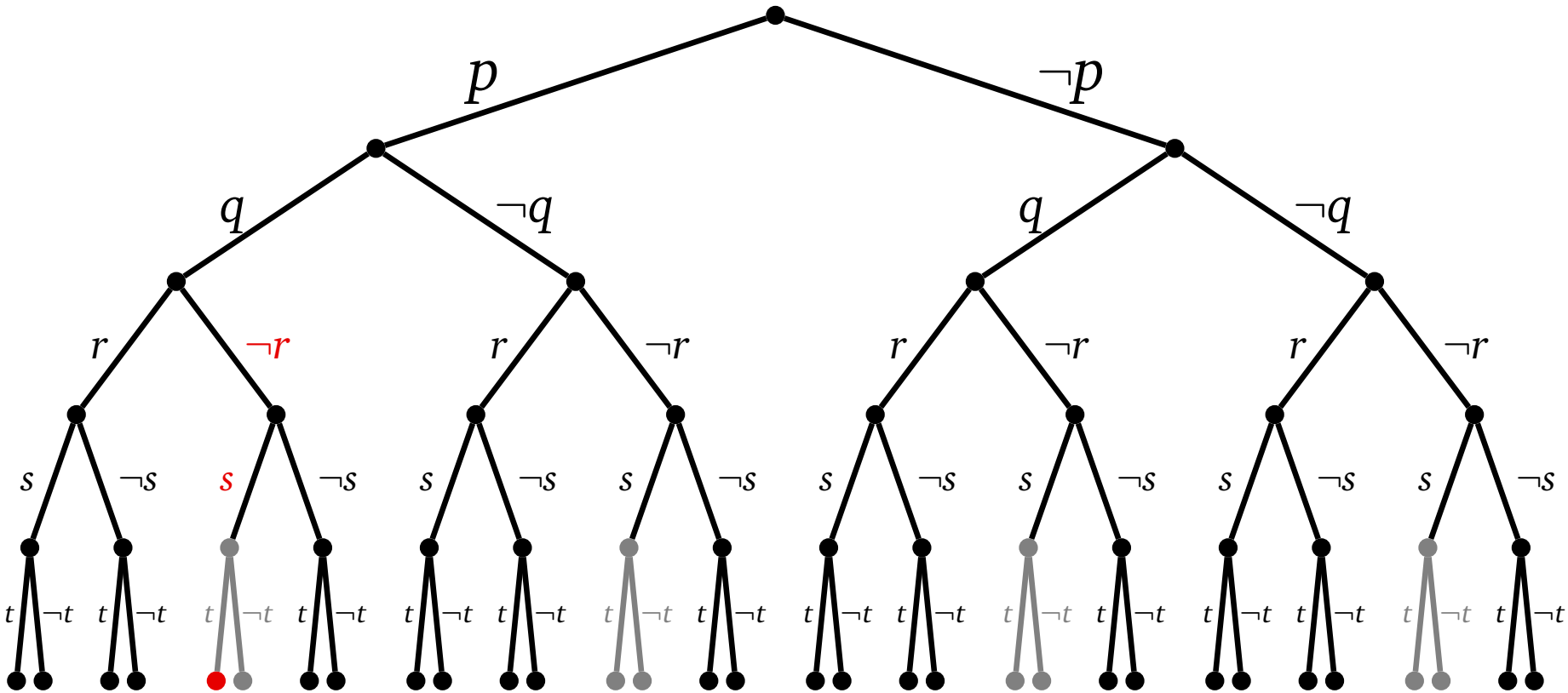


Conflict-Driven Clause Learning



Conflict caused by $\neg r$ and s! Add conflict clause $(r \vee \neg s)$.

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Implicitly prunes future subtrees.

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