NOTES ON LOANS AND SAVINGS

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1. Borrowing or saving money

We consider the situation where Jane has an initial sum of S_0 in her account, and at regular intervals, adds to that account a positive or negative amount (so in effect, adds to or removes money from that account), say Δ, with an interest of r applying to the time period between two successive operations. For i a strictly positive integer, let S_i denote the sum in Jane's account after i time periods. Then we have for all $i \in \mathbb{N}$:

$$S_i = S_0(1+r)^i + \Delta\Sigma_{j=0}^{i-1}(1+r)^j \tag{1}$$

as we immediately verify by induction: (1) trivially holds for $i = 0$, and given $i \in \mathbb{N}$, if (1) holds for i then

$$\begin{aligned} S_{i+1} &= S_i(1+r) + \Delta \\ &= \big(S_0(1+r)^i + \Delta\Sigma_{j=0}^{i-1}(1+r)^j\big)(1+r) + \Delta \\ &= S_0(1+r)^{i+1} + \Delta\Sigma_{j=1}^{i}(1+r)^j + \Delta \\ &= S_0(1+r)^{i+1} + \Delta\Sigma_{j=0}^{i}(1+r)^j \end{aligned}$$

2. Saving

In the case of an investment, then Δ is equal to S_0 and operations happen once a year. Assuming that after N years, Jane can close her account and decides to do so, then she would not add the last amount of Δ, so the final amount, the sum eventually available to her, say S, would be:

$$\begin{aligned} S &= S_N - \Delta \\ &= \Delta(1+r)^N + \Delta\Sigma_{j=0}^{N-1}(1+r)^j - \Delta \\ &= \Delta\Sigma_{j=1}^{N}(1+r)^j \\ &= \frac{\Delta}{r}\big((1+r)^{N+1} - (1+r)\big) \end{aligned}$$

We therefore have the following equations:

- $S = \frac{\Delta}{r}\big((1+r)^{N+1} - (1+r)\big)$
- $\Delta = \dfrac{Sr}{(1+r)^{N+1} - (1+r)}$
- $N = \dfrac{\log_{10}\Big(\big(\frac{Sr}{\Delta} + (1+r)\big)\Big)}{\log_{10}(1+r)} - 1$

3. Borrowing

In the case of a loan over N years, the operations happen once a month, and the final sum eventually becomes 0:

$$\begin{aligned} 0 = S_{12N} &= S_0(1+r)^{12N} + \Delta\Sigma_{j=0}^{12(N-1)}(1+r)^j \\ &= S_0(1+r)^{12N} + \frac{\Delta}{r}\big((1+r)^{12N} - 1\big) \end{aligned}$$

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We therefore have the following equations:

- $S_0 = -\dfrac{\Delta\big((1+r)^{12N}-1\big)}{r(1+r)^{12N}}$
- $\Delta = -\dfrac{S_0(1+r)^{12N}r}{(1+r)^{12N}-1}$
- $N = \dfrac{\log_{10}(\frac{\Delta}{rS_0+\Delta})}{12\log_{10}(1+r)}$

4. Effective interest rate

Let R be the annual interest rate. First a period, year, semester, quarter or month, is chosen. Let d be 1, 2, 4 or 12, respectively (note that 1, 2, 4 and 12 are the number of years, semesters, quarters and months in a year, respectively). The interest rate is first reduced to R/d and declared to be the interest for the chosen period, that is, the interest that has to be paid at the end of that period. This corresponds to an effective interest rate for the year, say $\widetilde{R}$, equal to $(1+R/d)^d-1$, which corresponds to an effective interest rate for the month equal to the number $\widehat{R}$ such that $(1+\widehat{R})^{12}=1+\widetilde{R}$, yielding $\widehat{R}=(1+\widetilde{R})^{\frac{1}{12}}-1$.

- For savings, $r=\widetilde{R}$, which is all the more advantageous to Jane that d is larger.
- For loans, $r=\widehat{R}$, which is all the more advantageous to Jane that d is smaller.