12. Exponential Time Hypothesis

COMP6741: Parameterized and Exact Computation

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Outline

1. SAT and k-SAT
2. Subexponential time algorithms
3. ETH and SETH
4. Algorithmic lower bounds based on ETH
5. Algorithmic lower bounds based on SETH
6. Further Reading
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SAT

Input: A propositional formula $F$ in conjunctive normal form (CNF)
Parameter: $n = |\text{var}(F)|$, the number of variables in $F$
Question: Is there an assignment to $\text{var}(F)$ satisfying all clauses of $F$?

$k$-SAT

Input: A CNF formula $F$ where each clause has length at most $k$
Parameter: $n = |\text{var}(F)|$, the number of variables in $F$
Question: Is there an assignment to $\text{var}(F)$ satisfying all clauses of $F$?

Example:

$$(x_1 \lor x_2) \land (\neg x_2 \lor x_3 \lor \neg x_4) \land (x_1 \lor x_4) \land (\neg x_1 \lor \neg x_3 \lor \neg x_4)$$
Algorithms for SAT

- Brute-force: \( O^*\left(2^n\right) \)
Algorithms for SAT

- Brute-force: \( O^*(2^n) \)
- ... after > 50 years of SAT solving
  (SAT association, SAT conference, JSAT journal, annual SAT competitions, ...)

(\[Calabro, Impagliazzo, Paturi, 2006\] \[Dantsin, Hirsch, 2009\])

However: no \( O^*(1.9999^n) \) time algorithm is known

Fastest known algorithms for 3-SAT:
- \( O^*(1.3071^n) \) randomized \[Hertli, 2014\]
- \( O^*(1.3303^n) \) deterministic \[Makino, Tamaki, Yamamoto, 2013\]

Could it be that 3-SAT cannot be solved in \( 2^{o(n)} \) time?

Could it be that SAT cannot be solved in \( O^*((2-\epsilon)n) \) time for any \( \epsilon > 0 \)?
Algorithms for SAT

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- ... after > 50 years of SAT solving
  (SAT association, SAT conference, JSAT journal, annual SAT competitions, ...)
- fastest known algorithm for SAT: $O^*(2^n(1-1/O(\log m/n)))$, where $m$ is the number of clauses [Calabro, Impagliazzo, Paturi, 2006] [Dantsin, Hirsch, 2009]
- However: no $O^*(1.9999^n)$ time algorithm is known
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- Could it be that 3-SAT cannot be solved in $2^{o(n)}$ time?
- Could it be that SAT cannot be solved in $O^*((2 - \epsilon)^n)$ time for any $\epsilon > 0$?
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Are there any NP-hard problems that can be solved in $2^{o(n)}$ time?
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Yes. For example, **Independent Set** is NP-complete even when the input graph is planar (can be drawn in the plane without edge crossings). Planar graphs have treewidth $O(\sqrt{n})$ and tree decompositions of that width can be found in polynomial time (“Planar separator theorem” [Lipton, Tarjan, 1979]). Using a tree decomposition based algorithm, **Independent Set** can be solved in $2^{O(\sqrt{n})}$ time on planar graphs.
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ETH and SETH

**Definition 1**
For each $k \geq 3$, define $\delta_k$ to be the infinimum of the set of constants $c$ such that $k$-SAT can be solved in $O^*(2^{c \cdot n})$ time.

**Conjecture 2 (Exponential Time Hypothesis (ETH))**
$\delta_3 > 0$.

**Conjecture 3 (Strong Exponential Time Hypothesis (SETH))**
$\lim_{k \to \infty} \delta_k = 1$.

**Notes:**
1. ETH $\Rightarrow$ 3-SAT cannot be solved in $2^{o(n)}$ time.
2. SETH $\Rightarrow$ SAT cannot be solved in $O^*((2 - \epsilon)^n)$ time for any $\epsilon > 0$.

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1The infinimum of a set of numbers is the largest number that is smaller or equal to each number in the set. E.g., the infinimum of $\{\varepsilon \in \mathbb{R} : \varepsilon > 0\}$ is 0.
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Suppose ETH is true

Can we infer lower bounds on the running time needed to solve other problems?
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Suppose there is a polynomial-time reduction from 3-SAT to a graph problem $\Pi$, which constructs an equivalent instance where the number of vertices of the output graph equals the number of variables of the input formula, $|V| = |\text{var}(F)|$.

Using the reduction, we can conclude that, if $\Pi$ has an $O^*(2^{o(|V|)})$ time algorithm, then 3-SAT has an $O^*(2^{o(|\text{var}(F)|)})$ time algorithm, contradicting ETH.

Therefore, we conclude that $\Pi$ has no $O^*(2^{o(|V|)})$ time algorithm unless ETH fails.
Sparsification Lemma

**Issue:** Many reductions from 3-SAT create a number of vertices / variables / elements that are related to the number of clauses of the 3-SAT instance.
Sparsification Lemma

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**Theorem 4 (Sparsification Lemma, [Impagliazzo, Paturi, Zane, 2001])**

For each $\varepsilon > 0$ and positive integer $k$, there is a $O^*(2^{\varepsilon n})$ time algorithm that takes as input a $k$-CNF formula $F$ with $n$ variables and outputs an equivalent formula $F' = \bigvee_{i=1}^{t} F_i$ that is a disjunction of $t \leq 2^{\varepsilon n}$ formulas $F_i$ with $\text{var}(F_i) = \text{var}(F)$ and $|\text{cla}(F_i)| = O(n)$. 
3-SAT with a linear number of clauses

Corollary 5

ETH ⇒ 3-SAT cannot be solved in $O^*(2^{o(n+m)})$ time where $m$ denotes the number of clauses of $F$.

Observation: Let $A$, $B$ be parameterized problems and $f$, $g$ be non-decreasing functions. Suppose there is a polynomial-parameter transformation from $A$ to $B$ such that if the parameter of an instance of $A$ is $k$, then the parameter of the constructed instance of $B$ is at most $g(k)$. Then an $O^*(2^{o(f(k))})$ time algorithm for $B$ implies an $O^*(2^{o(f(g(k)))})$ time algorithm for $A$. 
More general reductions are possible

**Definition 6 (SERF-reduction)**

A SubExponential Reduction Family from a parameterized problem $A$ to a parameterized problem $B$ is a family of Turing reductions from $A$ to $B$ (i.e., an algorithm for $A$, making queries to an oracle for $B$ that solves any instance for $B$ in constant time) for each $\varepsilon > 0$ such that

- for every instance $I$ for $A$ with parameter $k$, the running time is $O^*(2^{\varepsilon k})$, and
- for every query $I'$ to $B$ with parameter $k'$, we have that $k' \in O(k)$ and $|I'| = |I|^{O(1)}$.

**Note:** If $A$ is SERF-reducible to $B$ and $A$ has no $2^{o(k)}$ time algorithm, then $B$ has no $2^{o(k')}$ time algorithm.
Vertex Cover has no subexponential algorithm

For simplicity, assume all clauses have length 3.

3-CNF Formula $F = (u \lor v \lor \neg y) \land (\neg u \lor y \lor z) \land (\neg v \lor w \lor x) \land (x \lor y \lor \neg z)$

For a 3-CNF formula with $n$ variables and $m$ clauses, we create a Vertex Cover instance with $|V| = 2^n + 3m$ and $k = n + 2m$. 
Vertex Cover has no subexponential algorithm

Polynomial-parameter transformation from 3-SAT. For simplicity, assume all clauses have length 3.

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For a 3-CNF formula with $n$ variables and $m$ clauses, we create a Vertex Cover instance with $|V| = 2n + 3m$ and $k = n + 2m$. 
Vertex Cover has no subexponential algorithm II

**Theorem 7**

\[ \text{ETH} \Rightarrow \text{Vertex Cover has no } 2^{o(|V|)} \text{ time algorithm.} \]

**Theorem 8**

\[ \text{ETH} \Rightarrow \text{Vertex Cover has no } 2^{o(k)} \text{ time algorithm.} \]
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**Recall:** A *hitting set* of a set system $S = (V, H)$ is a subset $X$ of $V$ such that $X$ contains at least one element of each set in $H$, i.e., $X \cap Y \neq \emptyset$ for each $Y \in H$.

**elts-HITTING SET**

- **Input:** A set system $S = (V, H)$ and an integer $k$
- **Parameter:** $n = |V|$
- **Question:** Does $S$ have a hitting set of size at most $k$?
SETH-lower bound for Hitting Set

CNF Formula $F = (u \lor v \lor \neg y) \land (\neg u \lor y \lor z) \land (\neg v \lor w \lor x) \land (x \lor y \lor \neg z)$

Inidence graph of equivalent Hitting Set instance:

For a CNF formula with $n$ variables and $m$ clauses, we create a Hitting Set instance with $|V| = 2n$ and $k = n$. 
Theorem 9

\( \text{SETH} \Rightarrow \text{Hitting Set has no } O^*((2 - \varepsilon)|V|/2) \text{ time algorithm for any } \varepsilon > 0. \)

**Note:** With a more ingenious reduction, one can show that \( \text{Hitting Set} \) has no \( O^*((2 - \varepsilon)|V|) \) time algorithm for any \( \varepsilon > 0 \) under SETH.
A dominating set of a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that $N_G[S] = V$.

**vertex-DOMINATING SET**

- Input: A graph $G = (V, E)$ and an integer $k$
- Parameter: $n = |V|$
- Question: Does $G$ have a dominating set of size at most $k$?

- Prove that ETH $\Rightarrow$ vertex-DOMINATING SET has no $2^{o(n)}$ time algorithm.
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