Reasoning about (Lack of) Knowledge

Christoph Schwering

UNSW Sydney

COMP4418, Week 7

Motivation

John McCarthy (1927–2011):

- Stanford, MIT, Dartmouth
- Turing Award
- Invented Lisp (1958)
- Invented Garbage Collection (1959)
- Founding Father of AI (with Minsky, Newell, Simon, 1955)

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Advice Taker motivates (directly or indirectly) a lot of AI research, in particular what we'll be studying for the next three weeks

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Not only what we know is relevant, but also what we *don’t* know

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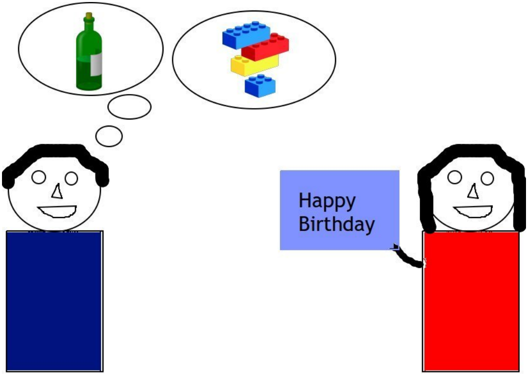

You don't know what's in the gift box.

You'll treat it with great care.

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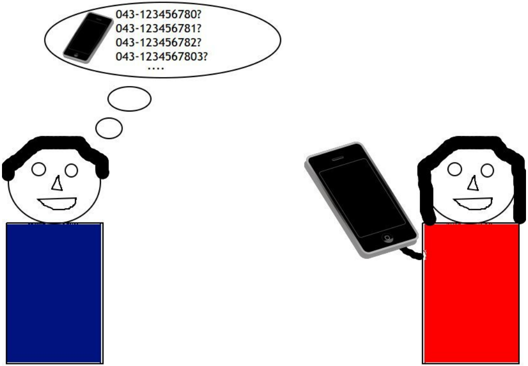

You know Jane has a phone, but you don't know her number.

You'll look it up.

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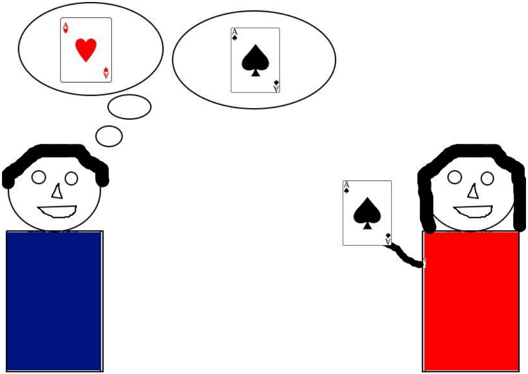

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You'll need a strategy that wins in either case.

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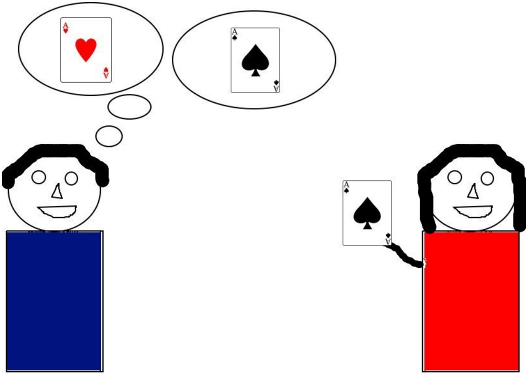

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How can we accurately **capture knowledge and non-knowledge**?

Overview of the Lecture

- **A Logic of Knowledge – The Propositional Fragment**
  - Why not classical logic?
  - Syntax and semantics
  - Omniscience, introspection, only-knowing
  - Representation theorem
- A Logic of Knowledge – The First-Order Case
- Extensions of the Logic of Knowledge

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- Usually: knowing is more than database lookup
  - $\alpha \in \mathrm{KB} \implies \alpha$ is explicit knowledge (= database lookup)
  - $\mathrm{KB} \models \alpha \implies \alpha$ is implicit knowledge (= logical inference)
  - Usually: explicit knowledge $\subsetneq$ (implicit) knowledge

Why Not Classical Logic?

Suppose all you know is $(p \vee q \vee r) \wedge (p \vee q \vee \neg r)$

Then:

1. You don't know that r.
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5. You don't know that q.

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<u>Problem</u>: Classical logic cannot express 6 directly in one formula

Why Not Classical Logic?

Suppose all you know is $(p \vee q \vee r) \wedge (p \vee q \vee \neg r) \wedge \neg k\_p \wedge \ldots$

Then:

1. You don't know that r. $\quad \mathrm{KB} \models \neg k\_r$
2. You don't know that $\neg r$. $\quad \mathrm{KB} \models \neg k\_not\_r$
3. You know that p or q. $\quad \mathrm{KB} \models k\_p\_or\_q$
4. You don't know that p. $\quad \mathrm{KB} \models \neg k\_p$
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6. You know that p or q, but not which.

$$\mathrm{KB} \models k\_p\_or\_q \wedge \neg k\_p \wedge \neg k\_p$$

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<u>Idea #1</u>: Compile $(p \vee q \vee c)$ to new atoms $k\_p, k\_p\_or\_q, \ldots$ ✗
Does not scale.

Why Not Classical Logic?

Suppose all you know is $(p \vee q \vee r) \wedge (p \vee q \vee \neg r)$

Then:

1. You don't know that r. $\quad \mathrm{KB} \models r = \mathrm{U}$
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$$\mathrm{KB} \models (p \vee q) \wedge p = \mathrm{U} \wedge q = \mathrm{U}$$

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Idea #2: Three-valued logic $\{0, 1, \mathrm{U}\}$ ✗
How would $\mathrm{U} \vee \mathrm{U}$ behave? Is it known? Unknown?

Why Not Classical Logic?

Suppose all you know is $(p \vee q \vee r) \wedge (p \vee q \vee \neg r)$

Then:

1. You don't know that r. $\quad \mathbf{O}\mathrm{KB} \models \neg\mathbf{K}r$
2. You don't know that $\neg r$. $\quad \mathbf{O}\mathrm{KB} \models \neg\mathbf{K}\neg r$
3. You know that p or q. $\quad \mathbf{O}\mathrm{KB} \models \mathbf{K}(p \vee q)$
4. You don't know that p. $\quad \mathbf{O}\mathrm{KB} \models \neg\mathbf{K}p$
5. You don't know that q. $\quad \mathbf{O}\mathrm{KB} \models \neg\mathbf{K}q$
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   $$\mathbf{O}\mathrm{KB} \models \mathbf{K}(p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q$$

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Idea #3: Add unary operators $\mathbf{O}$ and $\mathbf{K}$ to express knowledge ✓

The Language of $\mathcal{OL}_{\mathrm{PL}}$

The language of only-knowing (propositional fragment) $\mathcal{OL}_{\mathrm{PL}}$:

- $p, q, r, \ldots$ atomic propositions
- $\neg\alpha$ "not α"
- $(\alpha \vee \beta)$ "α or β"
- $(\alpha \wedge \beta)$ "α and β"
- $(\alpha \rightarrow \beta)$ "α implies β"
- $(\alpha \leftrightarrow \beta)$ "α is equivalent to β"
- $\mathbf{K}\alpha$ "α is known"
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Recap: Technical Terms (1)

A logical language is a *formal language* over an *alphabet* (here: $\{p, q, r, \ldots, (,), \neg, \vee, \mathbf{K}, \mathbf{O}\}$) and a *grammar* (previous slide), i.e., rules that allow us to phrase sentences in that language.

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The sentences carry *no meaning by themselves*. We define a *model theory* to give them a *semantics*, i.e., to define what sort of formal structure interprets a sentence. Such an *interpretation* I satisfies a sentence α, written $I \models \alpha$, or falsifies it, written $I \not\models \alpha$.

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A typical rule of a semantics is

$$I \models (\alpha \vee \beta) \text{ if and only if } I \models \alpha \text{ or } I \models \beta.$$

Note that $\vee$ is a symbol of the logical language, whereas "if and only if" and "or" are natural language expressions. The rule says that the symbol "$\vee$" corresponds to the natural language expression "or".

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We will sometimes take the liberty to omit brackets to ease readability. For instance, we write $(p \vee q \vee r)$ instead of $((p \vee q) \vee r)$ or $(p \vee (q \vee r))$, implicitly assuming our semantics of $\vee$ is associative.

Recap: Technical Terms (2)

The form of such an interpretation varies between logics. Propositional logic uses truth tables, first-order logic usually uses structures with a domain and interpretation function.

- When an interpretation I satisfies a sentence, we write $I \models \alpha$.
- When all interpretations satisfy a sentence α, then α is *valid* and we write $\models \alpha$.
- When all interpretations that satisfy some sentence Σ or set of sentences Σ also satisfy α, we say Σ *entails* α and write $\Sigma \models \alpha$.

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Different semantics are possible. What justifies a semantics? Typically there is a *proof theory*, and model theory and proof theory should be equivalent ($\models \alpha$ if and only if $\vdash \alpha$). Nevertheless, we will only focus on the semantics in the next weeks.

The Semantics of $\mathcal{OL}_{\mathrm{PL}}$

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- $e, w \models \mathbf{K}\alpha \iff$ for all worlds w', $w' \in e \Rightarrow e, w' \models \alpha$
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"$\Rightarrow$" stands for natural language expressions "only if".
"$\Leftrightarrow$" and "$\iff$" stand for natural language expressions "if and only if".

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Abbreviations

Recall:

- $(\alpha \wedge \beta) \stackrel{\text{def}}{=} \neg(\neg\alpha \vee \neg\beta)$
- $(\alpha \rightarrow \beta) \stackrel{\text{def}}{=} (\neg\alpha \vee \beta)$
- $(\alpha \leftrightarrow \beta) \stackrel{\text{def}}{=} (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$

$\wedge$ should be “and”
$\rightarrow$ should be “only if”
$\leftrightarrow$ should be “if and only if”.

Lemma: abbreviations

- $e, w \models \alpha \wedge \beta \iff e, w \models \alpha$ and $e, w \models \beta$
- $e, w \models \alpha \rightarrow \beta \iff e, w \models \alpha \Rightarrow e, w \models \beta$
- $e, w \models \alpha \leftrightarrow \beta \iff e, w \models \alpha \Leftrightarrow e, w \models \beta$

Proof on paper

Some Lemmas

Definition: objective, subjective

If ϕ mentions no atoms inside $\mathbf{K}$ or $\mathbf{O}$, we say ϕ is **objective**.
If σ mentions atoms only inside $\mathbf{K}$ or $\mathbf{O}$, we say σ is **subjective**.

- $((p \vee q) \wedge p \wedge q)$ is objective
- $\mathbf{K}((p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q)$ is subjective

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Lemma: objective, subjective

Let ϕ be objective. Then $e, w \models \phi \iff e', w \models \phi$.
Let σ be subjective. Then $e, w \models \sigma \iff e, w' \models \sigma$.

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- $((p \lor q) \land p \land q)$ is objective
- $\mathbf{K}((p \lor q) \land \neg \mathbf{K}p \land \neg \mathbf{K}q)$ is subjective

Lemma: objective, subjective

Let ϕ be objective. Then $e, w \models \phi \iff e', w \models \phi$.
Let σ be subjective. Then $e, w \models \sigma \iff e, w' \models \sigma$.

When ϕ is objective, "$w \models \phi$" stands for "for every e, $e, w \models \phi$".
When σ is subjective, "$e \models \sigma$" stands for "for every w, $e, w \models \sigma$".

Proof on paper

Examples

$$\boxed{e, w \models \mathbf{K}\alpha \iff \text{for all worlds } w',\ \ w \in e \Rightarrow e, w' \models \alpha}$$

Let $e \stackrel{\text{def}}{=} \{w \mid w \models (p \vee q \vee r) \wedge (p \vee q \vee \neg r)\}$

Examples

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Examples

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- $w \in e$

  $\iff \; w \models (p \vee q \vee r) \wedge (p \vee q \vee \neg r)$

Examples

$$e, w \models \mathbf{K}\alpha \iff \text{for all worlds } w',\ \ w \in e \Rightarrow e, w' \models \alpha$$

Let $e \stackrel{\text{def}}{=} \{w \mid w \models (p \vee q \vee r) \wedge (p \vee q \vee \neg r)\}$

- $w \in e$

$\iff\ w \models (p \vee q \vee r) \wedge (p \vee q \vee \neg r)$
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$\iff$ $w \models (p \vee q \vee r) \wedge (p \vee q \vee \neg r)$
$\iff$ $w \models (p \vee q \vee r)$ and $w \models (p \vee q \vee \neg r)$
$\iff$ $w[p] = 1$ or $w[q] = 1$ or $w[r] = 1$, and
$w[p] = 1$ or $w[q] = 1$ or $w[r] = 0$

Examples

$$e, w \models \mathbf{K}\alpha \iff \text{for all worlds } w', \; w \in e \Rightarrow e, w' \models \alpha$$

Let $e \stackrel{\text{def}}{=} \{w \mid w \models (p \vee q \vee r) \wedge (p \vee q \vee \neg r)\}$

- $w \in e$

$$\begin{aligned}
\iff\;& w \models (p \vee q \vee r) \wedge (p \vee q \vee \neg r) \\
\iff\;& w \models (p \vee q \vee r) \text{ and } w \models (p \vee q \vee \neg r) \\
\iff\;& w[p] = 1 \text{ or } w[q] = 1 \text{ or } w[r] = 1 \text{, and} \\
& w[p] = 1 \text{ or } w[q] = 1 \text{ or } w[r] = 0 \\
\iff\;& w[p] = 1 \text{ or } w[q] = 1
\end{aligned}$$

Examples

$$e, w \models \mathbf{K}\alpha \iff \text{for all worlds } w', \; w \in e \Rightarrow e, w' \models \alpha$$

Let $e \stackrel{\text{def}}{=} \{w \mid w \models (p \vee q \vee r) \wedge (p \vee q \vee \neg r)\}$

- $w \in e \iff w[p] = 1$ or $w[q] = 1$
- $e \models \mathbf{K}(p \vee q)$?

Examples

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Examples

$$e, w \models \mathbf{K}\alpha \iff \text{for all worlds } w', \;\; w \in e \Rightarrow e, w' \models \alpha$$

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$\iff$ for all w, $w \in e \Rightarrow w \models p$ or $w \models q$
$\iff$ for all w, $w \in e \Rightarrow w[p] = 1$ or $w[q] = 1$

Examples

$$\boxed{e, w \models \mathbf{K}\alpha \iff \text{for all worlds } w',\ \ w \in e \Rightarrow e, w' \models \alpha}$$

Let $e \stackrel{\text{def}}{=} \{w \mid w \models (p \lor q \lor r) \land (p \lor q \lor \neg r)\}$

- $w \in e \iff w[p] = 1$ or $w[q] = 1$
- $e \models \mathbf{K}(p \lor q)$?

$\iff$ for all w, $w \in e \Rightarrow w \models (p \lor q)$
$\iff$ for all w, $w \in e \Rightarrow w \models p$ or $w \models q$
$\iff$ for all w, $w \in e \Rightarrow w[p] = 1$ or $w[q] = 1$
$\iff$ for all w, $\big(w[p] = 1$ or $w[q] = 1\big) \Rightarrow \big(w[p] = 1$ or $w[q] = 1\big)$ ✓

Examples

$$e, w \models \mathbf{K}\alpha \iff \text{for all worlds } w',\ w \in e \Rightarrow e, w' \models \alpha$$

Let $e \stackrel{\text{def}}{=} \{w \mid w \models (p \vee q \vee r) \wedge (p \vee q \vee \neg r)\}$

- $w \in e \iff w[p] = 1$ or $w[q] = 1$
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  $\iff e \not\models \mathbf{K}p$

Examples

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$\iff$ for some w, $w \in e$ and $w[p] \neq 1$

$\iff$ for some w, $w[p] = 1$ or $w[q] = 1$, and $w[p] \neq 1$

Examples

$$e, w \models \mathbf{K}\alpha \iff \text{for all worlds } w', \; w \in e \Rightarrow e, w' \models \alpha$$

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- $e \models \neg\mathbf{K}p$?

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$\iff$ for some w, $w \in e$ and $w \not\models p$
$\iff$ for some w, $w \in e$ and $w[p] \neq 1$
$\iff$ for some w, $w[p] = 1$ or $w[q] = 1$, and $w[p] \neq 1$
$\iff$ for some w, $w[p] \neq 1$ and $w[q] = 1$ ✓

Examples

$$\boxed{e, w \models \mathbf{K}\alpha \iff \text{for all worlds } w',\ \ w \in e \Rightarrow e, w' \models \alpha}$$

Let $e \stackrel{\text{def}}{=} \{w \mid w \models (p \vee q \vee r) \wedge (p \vee q \vee \neg r)\}$

- $w \in e \iff w[p] = 1$ or $w[q] = 1$
- $e \models \mathbf{K}(p \vee q)$ ✓
- $e \models \neg\mathbf{K}p$ ✓
- $e \models \mathbf{K}(p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q$?

Examples

$$\boxed{e, w \models \mathbf{K}\alpha \iff \text{for all worlds } w', \; w \in e \Rightarrow e, w' \models \alpha}$$

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- $w \in e \iff w[p] = 1$ or $w[q] = 1$
- $e \models \mathbf{K}(p \vee q)$ ✓
- $e \models \neg\mathbf{K}p$ ✓
- $e \models \mathbf{K}(p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q$?
  $\iff$ $e \models \mathbf{K}(p \vee q)$ and $e \models \neg\mathbf{K}p$ and $e \models \neg\mathbf{K}q$ ✓

Examples

$$e, w \models \mathbf{K}\alpha \iff \text{for all worlds } w', \; w \in e \Rightarrow e, w' \models \alpha$$

Let $e \stackrel{\text{def}}{=} \{w \mid w \models (p \vee q \vee r) \wedge (p \vee q \vee \neg r)\}$

- $w \in e \iff w[p] = 1$ or $w[q] = 1$
- $e \models \mathbf{K}(p \vee q)$ ✓
- $e \models \neg\mathbf{K}p$ ✓
- $e \models \mathbf{K}(p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q$ ✓
- $e \models \mathbf{O}((p \vee q \vee r) \wedge (p \vee q \vee \neg r))$?

Examples

$$e, w \models \mathbf{K}\alpha \iff \text{for all worlds } w',\ w \in e \Rightarrow e, w' \models \alpha$$
$$e, w \models \mathbf{O}\alpha \iff \text{for all worlds } w',\ w \in e \Leftrightarrow e, w' \models \alpha$$

Let $e \stackrel{\text{def}}{=} \{w \mid w \models (p \vee q \vee r) \wedge (p \vee q \vee \neg r)\}$

- $w \in e \iff w[p] = 1$ or $w[q] = 1$
- $e \models \mathbf{K}(p \vee q)$ ✓
- $e \models \neg\mathbf{K}p$ ✓
- $e \models \mathbf{K}(p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q$ ✓
- $e \models \mathbf{O}((p \vee q \vee r) \wedge (p \vee q \vee \neg r))$?
  $\iff$ for all w, $w \in e \Leftrightarrow w \models ((p \vee q \vee r) \wedge (p \vee q \vee \neg r))$ ✓

Examples

$$e, w \models \mathbf{K}\alpha \iff \text{for all worlds } w',\ \ w \in e \Rightarrow e, w' \models \alpha$$
$$e, w \models \mathbf{O}\alpha \iff \text{for all worlds } w',\ \ w \in e \Leftrightarrow e, w' \models \alpha$$

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- $w \in e \iff w[p] = 1$ or $w[q] = 1$
- $e \models \mathbf{K}(p \vee q)$ ✓
- $e \models \neg\mathbf{K}p$ ✓
- $e \models \mathbf{K}(p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q$ ✓
- $e \models \mathbf{O}((p \vee q \vee r) \wedge (p \vee q \vee \neg r))$ ✓

Logical Omniscience

Logical omniscience means that an agent knows all the consequences of what they know. In particular, they know all valid sentences.

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Theorem: logical omniscience

If $\models \alpha \rightarrow \beta$, then $\models \mathbf{K}\alpha \rightarrow \mathbf{K}\beta$.
In particular: If $\models \alpha$, then $\models \mathbf{K}\alpha$.

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In particular: If $\models \alpha$, then $\models \mathbf{K}\alpha$.

Logical omniscience is often problematic:

- Philosophical problem: most agents are not omniscient
- Practical problem: omniscience makes reasoning intractable

We will look at methods to avoid these problems next week.

Proof on paper

Only-Knowing

The purpose of only-knowing is to capture a knowledge base.

Knowledge bases are usually objective.

The corresponding epistemic state is then unique:

Theorem: unique-model property

Let ϕ be objective. Then there is a unique e such that $e \models \mathbf{O}\phi$.

An entailment problem $\mathbf{O}\phi \models \mathbf{K}\alpha$ thus reduces to model checking:

$$e \models \mathbf{K}\alpha, \text{ where } e = \{w \mid w \models \phi\}?$$

Self-Knowledge

We can nest $\mathbf{K}$ operators to say that we know that we know.

Complete and accurate knowledge about own knowledge:

Theorem: positive and negative introspection

Positive introspection: $\models \mathbf{K}\alpha \rightarrow \mathbf{K}\mathbf{K}\alpha$
Negative introspection: $\models \neg\mathbf{K}\alpha \rightarrow \mathbf{K}\neg\mathbf{K}\alpha$

Why?
$e \models (\neg)\mathbf{K}\alpha \implies e, w \models (\neg)\mathbf{K}\alpha$ for all $w \in e \iff e, w \models \mathbf{K}(\neg)\mathbf{K}\alpha$.

Representation Theorem

Can we solve $\mathbf{O}\mathrm{KB} \models \alpha$ with ordinary, propositional reasoning?
That is, can we eliminate $\mathbf{K}$ and $\mathbf{O}$?
Then we could use standard reasoning system.

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Let KB, ϕ be objective. Then $\mathbf{O}\mathrm{KB} \models \mathbf{K}\phi \iff \mathrm{KB} \models \phi$.

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Ex.: Let $\mathrm{KB} \stackrel{\text{def}}{=} (p \vee q \vee r) \wedge (p \vee q \vee \neg r)$.

$$\mathbf{O}\mathrm{KB} \models \mathbf{K}\big((p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q\big)?$$

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$$\mathbf{O}\mathrm{KB} \models \mathbf{K}\big((p \vee q) \wedge \neg \underbrace{\mathbf{K}p}_{\mathrm{KB}\models p?\ ✗} \wedge \neg \underbrace{\mathbf{K}q}_{\mathrm{KB}\models q?\ ✗}\big)?$$

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Ex.: Let $\mathrm{KB} \stackrel{\text{def}}{=} (p \vee q \vee r) \wedge (p \vee q \vee \neg r)$.

$$\mathbf{O}\mathrm{KB} \models \mathbf{K}\big((p \vee q) \wedge \neg\text{FALSE} \wedge \neg\text{FALSE}\big)?$$

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Let KB, ϕ be objective. Then $\mathbf{O}\mathrm{KB} \models \mathbf{K}\phi \iff \mathrm{KB} \models \phi$.

Idea: replace nested $\mathbf{K}\phi$ with TRUE if $\mathrm{KB} \models \phi$, otherwise with FALSE.

Ex.: Let $\mathrm{KB} \stackrel{\text{def}}{=} (p \lor q \lor r) \land (p \lor q \lor \neg r)$.

$$\mathrm{KB} \models \big((p \lor q) \land \neg\text{FALSE} \land \neg\text{FALSE}\big)? \quad ✓$$

Representation Theorem

Can we solve $\mathbf{O}\mathrm{KB} \models \alpha$ with ordinary, propositional reasoning?
That is, can we eliminate $\mathbf{K}$ and $\mathbf{O}$?
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Theorem

Let KB, ϕ be objective. Then $\mathbf{O}\mathrm{KB} \models \mathbf{K}\phi \iff \mathrm{KB} \models \phi$.

Idea: replace nested $\mathbf{K}\phi$ with TRUE if $\mathrm{KB} \models \phi$, otherwise with FALSE.

Ex.: Let $\mathrm{KB} \stackrel{\text{def}}{=} (p \vee q \vee r) \wedge (p \vee q \vee \neg r)$.

$\mathrm{KB} \models \big((p \vee q) \wedge \neg\text{FALSE} \wedge \neg\text{FALSE}\big)$? ✓

Next slide formalises this idea.

Sneak preview: It'll become more difficult in the first-order case:
What would you replace $\mathbf{K}Q(x)$ with in $\mathbf{K}\exists x\,(P(x) \wedge \neg\mathbf{K}Q(x))$? We'll see later.

Representation Theorem (2)

Definition: representation operators

For objective KB and ϕ, let $\text{RES}[\text{KB}, \phi] \stackrel{\text{def}}{=} \begin{cases} \text{TRUE} & \text{if } \text{KB} \models \phi \\ \text{FALSE} & \text{otherwise} \end{cases}$

where TRUE is some tautology (e.g., $p \vee \neg p$) and FALSE is $\neg$TRUE.

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where TRUE is some tautology (e.g., $p \vee \neg p$) and FALSE is $\neg$TRUE.

- $\|P\|_{\mathrm{KB}} \stackrel{\text{def}}{=} P$
- $\|\neg\alpha\|_{\mathrm{KB}} \stackrel{\text{def}}{=} \neg\|\alpha\|_{\mathrm{KB}}$
- $\|(\alpha \vee \beta)\|_{\mathrm{KB}} \stackrel{\text{def}}{=} (\|\alpha\|_{\mathrm{KB}} \vee \|\beta\|_{\mathrm{KB}})$
- $\|\mathbf{K}\alpha\|_{\mathrm{KB}} \stackrel{\text{def}}{=} \mathrm{RES}[\mathrm{KB}, \|\alpha\|_{\mathrm{KB}}]$

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- $\|\mathbf{K}\alpha\|_{\mathrm{KB}} \stackrel{\text{def}}{=} \mathrm{RES}[\mathrm{KB}, \|\alpha\|_{\mathrm{KB}}]$

Theorem: representation theorem

$\mathbf{O}\mathrm{KB} \models \alpha \iff \models \|\alpha\|_{\mathrm{KB}}$.

Example

$$\|\mathbf{K}\alpha\|_{\mathrm{KB}} \stackrel{\text{def}}{=} \text{RES}[\mathrm{KB}, \|\alpha\|_{\mathrm{KB}}]$$

$$\text{RES}[\mathrm{KB}, \phi] \stackrel{\text{def}}{=} \text{“}\mathrm{KB} \models \phi\text{?”}$$

Let $\mathrm{KB} \stackrel{\text{def}}{=} (p \vee q \vee r) \wedge (p \vee q \vee \neg r)$.

$\mathbf{O}\mathrm{KB} \models \mathbf{K}\big((p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q\big)$?

Example

$$\|\mathbf{K}\alpha\|_{\mathrm{KB}} \stackrel{\text{def}}{=} \text{RES}[\mathrm{KB}, \|\alpha\|_{\mathrm{KB}}]$$

$$\text{RES}[\mathrm{KB}, \phi] \stackrel{\text{def}}{=} \text{“}\mathrm{KB} \models \phi?\text{”}$$

Let $\mathrm{KB} \stackrel{\text{def}}{=} (p \vee q \vee r) \wedge (p \vee q \vee \neg r)$.

$$\mathbf{O}\mathrm{KB} \models \mathbf{K}\big((p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q\big)$$

$$\iff \models \|\mathbf{K}\big((p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q\big)\|_{\mathrm{KB}}$$

Example

$$\|\mathbf{K}\alpha\|_{\mathrm{KB}} \stackrel{\text{def}}{=} \mathsf{RES}[\mathrm{KB}, \|\alpha\|_{\mathrm{KB}}]$$
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$\mathbf{O}\mathrm{KB} \models \mathbf{K}\big((p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q\big)$

$\iff \models \|\mathbf{K}\big((p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q\big)\|_{\mathrm{KB}}$

$\iff \models \mathsf{RES}[\mathrm{KB}, \|\big((p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q\big)\|_{\mathrm{KB}}]$

Example

$$\|\mathbf{K}\,\alpha\|_{\mathrm{KB}} \stackrel{\text{def}}{=} \mathsf{RES}[\mathrm{KB}, \|\alpha\|_{\mathrm{KB}}]$$
$$\mathsf{RES}[\mathrm{KB}, \phi] \stackrel{\text{def}}{=} \text{“KB} \models \phi\text{?”}$$

Let $\mathrm{KB} \stackrel{\text{def}}{=} (p \vee q \vee r) \wedge (p \vee q \vee \neg r)$.

$\mathbf{O}\mathrm{KB} \models \mathbf{K}\big((p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q\big)$

$$\iff \quad \models \|\mathbf{K}\big((p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q\big)\|_{\mathrm{KB}}$$
$$\iff \quad \models \mathsf{RES}[\mathrm{KB}, \|\big((p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q\big)\|_{\mathrm{KB}}]$$
$$\iff \quad \models \mathsf{RES}[\mathrm{KB}, \big((p \vee q) \wedge \neg \underbrace{\|\mathbf{K}p\|_{\mathrm{KB}}}_{\mathrm{KB}\models p?} \wedge \neg \underbrace{\|\mathbf{K}q\|_{\mathrm{KB}}}_{\mathrm{KB}\models q?}\big)]$$

Example

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$$\mathsf{RES}[\mathrm{KB}, \phi] \stackrel{\mathrm{def}}{=} \text{“}\mathrm{KB} \models \phi\text{?”}$$

Let KB $\stackrel{\mathrm{def}}{=} (p \vee q \vee r) \wedge (p \vee q \vee \neg r)$.

$$\mathbf{O}\mathrm{KB} \models \mathbf{K}\big((p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q\big)$$

$$\iff \models \|\mathbf{K}\big((p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q\big)\|_{\mathrm{KB}}$$

$$\iff \models \mathsf{RES}[\mathrm{KB}, \|\big((p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q\big)\|_{\mathrm{KB}}]$$

$$\iff \models \mathsf{RES}[\mathrm{KB}, \big((p \vee q) \wedge \neg \underbrace{\|\mathbf{K}p\|_{\mathrm{KB}}}_{\mathrm{KB}\models p?} \wedge \neg \underbrace{\|\mathbf{K}q\|_{\mathrm{KB}}}_{\mathrm{KB}\models q?}\big)]$$

$$\iff \models \underbrace{\mathsf{RES}[\mathrm{KB}, \big((p \vee q) \wedge \neg\mathsf{FALSE} \wedge \neg\mathsf{FALSE}\big)]}_{\mathrm{KB}\models(p\vee q)\wedge\neg\mathsf{FALSE}\wedge\neg\mathsf{FALSE}?}$$

Example

$$\|\mathbf{K}\,\alpha\|_{\mathrm{KB}} \stackrel{\text{def}}{=} \mathsf{RES}[\mathrm{KB}, \|\alpha\|_{\mathrm{KB}}]$$
$$\mathsf{RES}[\mathrm{KB}, \phi] \stackrel{\text{def}}{=} \text{“}\mathrm{KB} \models \phi?\text{”}$$

Let $\mathrm{KB} \stackrel{\text{def}}{=} (p \vee q \vee r) \wedge (p \vee q \vee \neg r)$.

$$\mathbf{O}\mathrm{KB} \models \mathbf{K}\big((p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q\big)$$

$$\iff \models \|\mathbf{K}\big((p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q\big)\|_{\mathrm{KB}}$$

$$\iff \models \mathsf{RES}[\mathrm{KB}, \|\big((p \vee q) \wedge \neg\mathbf{K}p \wedge \neg\mathbf{K}q\big)\|_{\mathrm{KB}}]$$

$$\iff \models \mathsf{RES}[\mathrm{KB}, \big((p \vee q) \wedge \neg \underbrace{\|\mathbf{K}p\|_{\mathrm{KB}}}_{\mathrm{KB}\models p?} \wedge \neg \underbrace{\|\mathbf{K}q\|_{\mathrm{KB}}}_{\mathrm{KB}\models q?}\big)]$$

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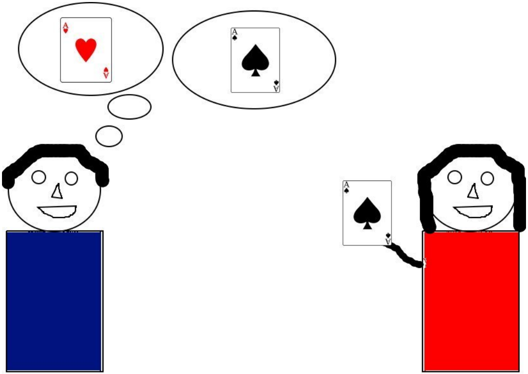
$$\iff \models \mathsf{TRUE} \quad \checkmark$$

Overview of the Lecture

- A Logic of Knowledge – The Propositional Fragment
- **A Logic of Knowledge – The First-Order Case**
  - Why first-order logic?
  - Syntax and semantics
  - Knowing that vs knowing what
  - Representation theorem
- Extensions of the Logic of Knowledge

Why Is $\mathcal{OL}_{\mathrm{PL}}$ Not Enough?



$$\mathbf{K}\big((\spadesuit \vee \heartsuit) \wedge \neg\mathbf{K}\spadesuit \wedge \neg\mathbf{K}\heartsuit\big)$$

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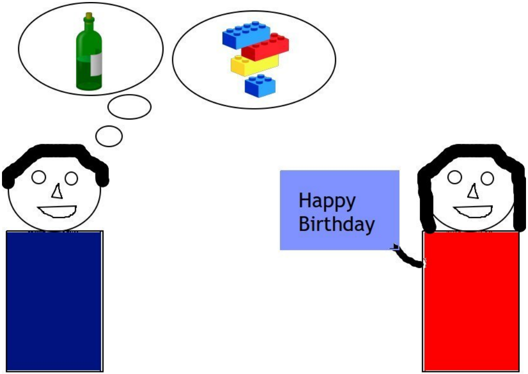

$$\mathbf{K}\exists x\,\big(\mathrm{InBox}(x) \land \neg\mathbf{K}\,\mathrm{InBox}(x)\big)$$

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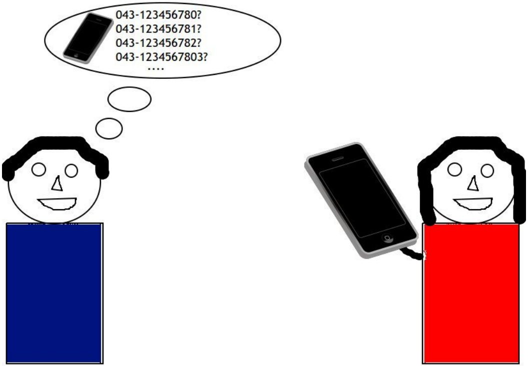

$$\mathbf{K}\exists x\left(\text{numberOf}(\text{Jane}) = x \wedge \neg\mathbf{K}\text{numberOf}(\text{Jane}) = x\right)$$

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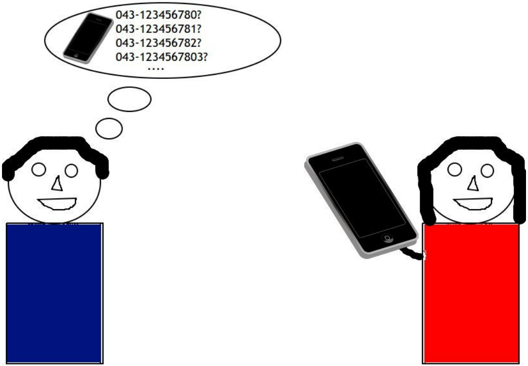

$$\mathbf{K}\exists x\left(\mathrm{numberOf}(\mathrm{Jane})=x\wedge\neg\mathbf{K}\,\mathrm{numberOf}(\mathrm{Jane})=x\right)$$

"all" or "some" $\Longrightarrow$ first-order quantification

The Language of $\mathcal{OL}$

Terms:

- $x, x', x_1, x_2, \ldots$ — first-order variables
- $^{\#}1, {}^{\#}2, {}^{\#}3, \ldots$ — standard names
- $f(t_1, \ldots, t_j)$ — functions

Formulas:

- $P(t_1, \ldots, t_j)$ — atomic formulas
- $t_1 = t_2$ — equality expressions
- $\exists x\, \alpha$ — "for some x, α"
- $\forall x\, \alpha$ — "for all x, α"
- $\neg\alpha \quad (\alpha \vee \beta) \quad (\alpha \wedge \beta) \quad (\alpha \rightarrow \beta) \quad (\alpha \leftrightarrow \beta) \quad \mathbf{K}\alpha \quad \mathbf{O}\alpha$

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- $\forall x\, \alpha \overset{\text{def}}{=} \neg\exists x\, \neg\alpha$ "for all x, α"
- $\neg\alpha \quad (\alpha \vee \beta) \quad (\alpha \wedge \beta) \quad (\alpha \rightarrow \beta) \quad (\alpha \leftrightarrow \beta) \quad \mathbf{K}\alpha \quad \mathbf{O}\alpha$

Why Standard Names?

- Consider in classical logic:
  $\mathrm{fatherOf}(\mathrm{Sally}) = \mathrm{bestFriend}(\mathrm{Jane}) \wedge$
  $\mathrm{fatherOf}(\mathrm{Sally}) = \mathrm{bossOf}(\mathrm{John})$
  - Who is father of Sally?
  - "Jane's best friend" is not a good answer
  - "John's boss" is not a good answer
  - Classical logic offers no way of identifying him
  - Reason: interpretations $\langle D, \Phi \rangle$ have different domains

- Standard names correspond to an implicit infinite domain

- Standard names allow to *identify* individuals in formulas:
  $\mathrm{fatherOf}(\mathrm{Sally}) = \mathrm{Frank}$

The Semantics of $\mathcal{OL}$ (1)

Definition: semantics of $\mathcal{OL}$ (1)

$f(\vec{n})$ or $P(\vec{n})$ are **primitive** iff all n_i are standard names.
A term or a formula is **ground** iff it mentions no variable.

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The **denotation** of a ground term w.r.t. w is defined as

- $w(n) \stackrel{\text{def}}{=} n$ for every standard name n
- $w(f(t_1, \ldots, t_j)) \stackrel{\text{def}}{=} w[f(w(t_1), \ldots, w(t_j))]$

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E.g., if Frank is Mia's father and Mia is Jane's mother:

$$
\begin{aligned}
& w(\text{fatherOf}(\text{motherOf}(\text{Jane}))) \\
= \; & w[\text{fatherOf}(w[\text{motherOf}(\text{Jane})])] \\
= \; & w[\text{fatherOf}(\text{Mia})] \\
= \; & \text{Frank}.
\end{aligned}
$$

The Semantics of $\mathcal{OL}$ (2)

Definition: semantics of $\mathcal{OL}$

An **epistemic state** e is a set of worlds.

- $e, w \models P(t_1, \ldots, t_j) \iff w[P(w(t_1), \ldots, w(t_j))] = 1$
- $e, w \models t_1 = t_2 \iff w(t_1) = w(t_2)$

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- $e, w \models \neg\alpha \iff e, w \not\models \alpha$
- $e, w \models (\alpha \lor \beta) \iff e, w \models \alpha$ or $e, w \models \beta$

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- $e, w \models \exists x\, \alpha \iff e, w \models \alpha^x_n$ for some standard name n

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- $e, w \models \mathbf{K}\alpha \iff$ for all worlds w', $w' \in e \Rightarrow e, w' \models \alpha$
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Knowing That vs Knowing What

- $\mathbf{K}\exists x\,\mathrm{Secret}(x)$ I know *that* some x is a secret
- $\exists x\,\mathbf{K}\mathrm{Secret}(x)$ I know *which* x is a secret

- $\mathbf{K}\exists x\,\mathrm{fatherOf}(\mathrm{Sally}) = x$ I know *that* Sally has a father
- $\exists x\,\mathbf{K}\mathrm{fatherOf}(\mathrm{Sally}) = x$ I know *who* Sally's father is

- $\mathbf{K}\exists x\,\alpha$ = *de dicto* knowledge
- $\exists x\,\mathbf{K}\,\alpha$ = *de re* knowledge

Theorem: quantifying-in

$\models \forall x\,\mathbf{K}\,\alpha \leftrightarrow \mathbf{K}\forall x\,\alpha$
$\models \exists x\,\mathbf{K}\,\alpha \rightarrow \mathbf{K}\exists x\,\alpha$
$\not\models \mathbf{K}\exists x\,\alpha \rightarrow \exists x\,\mathbf{K}\,\alpha$

Some Properties Inherited From $\mathcal{OL}_{\mathrm{PL}}$

Definition: subjective, objective

If ϕ mentions no fun/pred inside $\mathbf{K}$ or $\mathbf{O}$, we say ϕ is **objective**.
If σ mentions fun/pred only inside $\mathbf{K}$ or $\mathbf{O}$, we say σ is **subjective**.

Theorem: logical omniscience

If $\models \alpha \rightarrow \beta$, then $\models \mathbf{K}\alpha \rightarrow \mathbf{K}\beta$.
If $\models \alpha$, then $\models \mathbf{K}\alpha$.

Theorem: unique-model property

Let ϕ be objective. Then there is a unique e such that $e \models \mathbf{O}\phi$.

Theorem: positive and negative introspection

Positive introspection: $\models \mathbf{K}\alpha \rightarrow \mathbf{K}\mathbf{K}\alpha$
Negative introspection: $\models \neg\mathbf{K}\alpha \rightarrow \mathbf{K}\neg\mathbf{K}\alpha$

Example

$$e, w \models \exists x\, \alpha \iff e, w \models \alpha^x_n \text{ for some standard name } n$$
$$e, w \models \mathbf{K}\alpha \iff \text{for all worlds } w',\ w \in e \Rightarrow e, w' \models \alpha$$
$$e, w \models \mathbf{O}\alpha \iff \text{for all worlds } w',\ w \in e \Leftrightarrow e, w' \models \alpha$$

Let $\mathrm{KB} \stackrel{\text{def}}{=} \exists x\, (x \neq {}^{\#}1 \wedge P(x))$

Example

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- $e \models \mathbf{O}\mathrm{KB}$

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$\iff \; w \in e \Leftrightarrow w \models \exists x\, (x \neq {}^{\#}1 \wedge P(x))$

$\iff \; w \in e \Leftrightarrow w[P(n)] = 1$ for some $n \in \{{}^{\#}2, {}^{\#}3, \ldots\}$

Example

$$e, w \models \exists x\, \alpha \iff e, w \models \alpha^x_n \text{ for some standard name } n$$
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- $e \models \mathbf{K}\exists x\,(P(x) \wedge \neg\mathbf{K}P(x))$

Example

$$e, w \models \exists x\, \alpha \iff e, w \models \alpha^x_n \text{ for some standard name } n$$
$$e, w \models \mathbf{K}\alpha \iff \text{for all worlds } w',\, w \in e \Rightarrow e, w' \models \alpha$$
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- $e \models \mathbf{K}\exists x\,(P(x) \wedge \neg\mathbf{K}P(x))$

$\iff \quad$ for all w, $\; w \in e \Rightarrow$ for some n, $\; e, w \models P(n) \wedge \neg\mathbf{K}P(n)$

Example

$$e, w \models \exists x\, \alpha \iff e, w \models \alpha^x_n \text{ for some standard name } n$$
$$e, w \models \mathbf{K}\alpha \iff \text{for all worlds } w',\ w \in e \Rightarrow e, w' \models \alpha$$
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- $e \models \mathbf{O}\mathrm{KB}$

$\iff$ $w \in e \Leftrightarrow w \models \exists x\, (x \neq {}^{\#}1 \wedge P(x))$

$\iff$ $w \in e \Leftrightarrow w[P(n)] = 1$ for some $n \in \{{}^{\#}2, {}^{\#}3, \ldots\}$

- $e \models \mathbf{K}\exists x\, (P(x) \wedge \neg\mathbf{K}P(x))$

$\iff$ for all w, $w \in e \Rightarrow$ for some n, $e, w \models P(n) \wedge \neg\mathbf{K}P(n)$

$\iff$ for all w, $w \in e \Rightarrow$ for some n, $e, w \models P(n)$ and $e, w \models \neg\mathbf{K}P(n)$

Example

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$\iff$ for all w, $w \in e \Rightarrow$ for some n, $e, w \models P(n)$ and
for some w', $w' \in e$ and $e, w' \not\models P(n)$

$\iff$ for all w, $w \in e \Rightarrow$ for some n, $w[P(n)] = 1$ and
for some w', $w' \in e$ and $w'[P(n)] \neq 1$

Comparison with Tarski Semantics

- Traditional FOL semantics
  - Interpretation $\langle D, \Phi \rangle$ plus variable mapping μ
  - $\langle D, \Phi \rangle, \mu \models P(t_1, \ldots, t_j) \iff \langle d_1, \ldots, d_j \rangle \in \Phi(P)$
    where $d_i = \langle D, \Phi \rangle, \mu \| t_i \|$
  - $\langle D, \Phi \rangle, \mu \models \exists x\, \alpha \iff \langle D, \Phi \rangle, \mu^x_d \models \alpha$ for some $d \in D$
  - Purpose: reason about mathematics
  - Disadvantage: cumbersome to work with

- Our semantics
  - World maps primitive functions to names, predicates to $\{0, 1\}$
  - $w \models P(t_1, \ldots, t_j) \iff w[P(n_1, \ldots, n_j)] = 1$ where $n_i = w(t_i)$
  - $w \models \exists x\, \alpha \iff w \models \alpha^x_n$ for some standard name n
  - Purpose: reason about knowledge
  - Disadvantage: domain is always countably infinite
    - $\forall x\, (x = t_1 \vee \ldots \vee x = t_j)$ asserts finite domain in classical FOL
    - $\forall x\, (x = t_1 \vee \ldots \vee x = t_j)$ is unsatisfiable in $\mathcal{OL}$
    - but can be simulated with predicate: $\forall x\, (P(x) \leftrightarrow (x = t_1 \vee \ldots \vee x = t_j))$
    - classical FOL cannot distinguish countably infinite from uncountably infinite domains anyway

Representation Theorem (1)

$$\mathbf{O}\mathrm{KB} \models \exists x\, \mathbf{K} P(x)?$$

How can we represent the known instances of an objective formula?

- $\mathrm{KB} \stackrel{\text{def}}{=} (P(\#1) \land P(\#2))$ — $\#1, \#2$ are known P-instances
- $\mathrm{KB} \stackrel{\text{def}}{=} (P(\#1) \lor P(\#2))$ — no known P-instances
- $\mathrm{KB} \stackrel{\text{def}}{=} \forall x P(x)$ — all names are known P-instances
- $\mathrm{KB} \stackrel{\text{def}}{=} \forall x\, (x \neq \#1 \to P(x))$ — $\#2, \#3, \ldots$ are known P-instances
- $\mathrm{KB} \stackrel{\text{def}}{=} (Q(\#1) \land \forall x\, (Q(x) \to P(x))$ — $\#1$ is known P-instance

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- $\mathrm{KB} \stackrel{\text{def}}{=} (Q(\#1) \wedge \forall x\,(Q(x) \rightarrow P(x))$ — $\#1$ is known P-instance

Let $n_1, \ldots, n_j$ be names in KB and let n' be a new one.

$$\begin{aligned}\text{RES}[\mathrm{KB}, P(x)] \;\stackrel{\text{def}}{=}\;& (x = n_1 \wedge \text{“}\mathrm{KB} \models P(n_1)\text{”}) \vee \\ & \ldots \\ & (x = n_j \wedge \text{“}\mathrm{KB} \models P(n_j)\text{”}) \vee \\ & (x \neq n_1 \wedge \ldots \wedge x \neq n_j \wedge \text{“}\mathrm{KB} \models P(n')\text{”})\end{aligned}$$

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How can we represent the known instances of an objective formula?

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- $\mathrm{KB} \stackrel{\text{def}}{=} (P(\#1) \vee P(\#2))$ $\qquad$ FALSE
- $\mathrm{KB} \stackrel{\text{def}}{=} \forall x P(x)$ $\qquad$ TRUE
- $\mathrm{KB} \stackrel{\text{def}}{=} \forall x\,(x \neq \#1 \rightarrow P(x))$ $\qquad x \neq \#1$
- $\mathrm{KB} \stackrel{\text{def}}{=} (Q(\#1) \wedge \forall x\,(Q(x) \rightarrow P(x))$ $\qquad x = \#1$

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& \ldots \\
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- $\mathrm{KB} \stackrel{\text{def}}{=} \forall x P(x)$ — TRUE
- $\mathrm{KB} \stackrel{\text{def}}{=} \forall x\,(x \neq \#1 \rightarrow P(x))$ — $x \neq \#1$
- $\mathrm{KB} \stackrel{\text{def}}{=} (Q(\#1) \wedge \forall x\,(Q(x) \rightarrow P(x))$ — $x = \#1$

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& (x \neq n_1 \wedge \ldots \wedge x \neq n_j \wedge \text{“}\mathrm{KB} \models P(n')\text{”})
\end{aligned}$$

Representation Theorem (2)

Definition: representation of known instances

If ϕ has a free variable x and $n_1, \ldots, n_j$ are the names mentioned in KB, ϕ, and n' is a new name:

$$\begin{aligned}\mathsf{RES}[\mathrm{KB}, \phi] \stackrel{\text{def}}{=} \; & (x = n_1 \wedge \mathsf{RES}[\mathrm{KB}, \alpha^x_{n_1}]) \vee \\ & \ldots \\ & (x = n_j \wedge \mathsf{RES}[\mathrm{KB}, \alpha^x_{n_j}]) \vee \\ & (x \neq n_1 \wedge \ldots \wedge x \neq n_j \wedge \mathsf{RES}[\mathrm{KB}, \phi^x_{n'}]^{n'}_x)\end{aligned}$$

If ϕ has no free variables:

$$\mathsf{RES}[\mathrm{KB}, \phi] \stackrel{\text{def}}{=} \begin{cases} \mathsf{TRUE} & \text{if } \mathrm{KB} \models \phi \\ \mathsf{FALSE} & \text{otherwise} \end{cases}$$

Representation Theorem (3)

$\|\cdot\|$ operator gets a rule for $\exists x\,\alpha$:

Definition: representation operators

- $\|\phi\|_{\mathrm{KB}} \stackrel{\text{def}}{=} \phi$ for objective ϕ
- $\|\neg\alpha\|_{\mathrm{KB}} \stackrel{\text{def}}{=} \neg\|\alpha\|_{\mathrm{KB}}$
- $\|(\alpha \vee \beta)\|_{\mathrm{KB}} \stackrel{\text{def}}{=} (\|\alpha\|_{\mathrm{KB}} \vee \|\beta\|_{\mathrm{KB}})$
- $\|\exists x\,\alpha\|_{\mathrm{KB}} \stackrel{\text{def}}{=} \exists x\,\|\alpha\|_{\mathrm{KB}}$
- $\|\mathbf{K}\alpha\|_{\mathrm{KB}} \stackrel{\text{def}}{=} \mathrm{RES}[\mathrm{KB}, \|\alpha\|_{\mathrm{KB}}]$

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Theorem: representation theorem

$\mathbf{O}\mathrm{KB} \models \alpha \iff \models \|\alpha\|_{\mathrm{KB}}.$

Overview of the Lecture

- A Logic of Knowledge – The Propositional Fragment

- A Logic of Knowledge – The First-Order Case

- **Extensions of the Logic of Knowledge**
  - Multiple agents
  - Probabilities
  - Conditional belief
  - Limited Belief (week 8)
  - Actions (week 9)

Multi-Agent Belief

Mike does not know what is in the gift box, but he knows that Jane knows what is in there:

$$\mathbf{K}_{\text{Mike}} \exists x \left(\text{InBox}(x) \land \neg \mathbf{K}_{\text{Mike}} \text{InBox}(x) \land \mathbf{K}_{\text{Jane}} \text{InBox}(x) \right)$$

Epistemic states get more complex: in every possible world, Mike considers a whole set of worlds to be possible from Jane's perspective.

Probabilities

I believe that with probability .999, there is no bomb in the gift box:

$$\mathbf{B}\left(\neg\exists x\left(\mathrm{InBox}(x) \wedge \mathrm{Bomb}(x)\right) : 0.999\right)$$

An epistemic state is now probability distribution over possible worlds.

Conditional Belief

I believe that if something is in the gift box, it's probably not a bomb:

$$\mathbf{B}\,(\exists x\,\mathrm{InBox}(x) \Rightarrow \neg\mathrm{Bomb}(x))$$

Epistemic state ranks possible worlds by plausibility and checks if the most-plausible worlds where $\exists x\,\mathrm{InBox}(x)$ is true also satisfy $\mathrm{Bomb}(x)$.

- A knowledge base is now a collection of conditionals
  "if _____, then most likely _____"
- What sort of ranking should these conditionals induce?