# Noncooperative Games

# COMP4418 Knowledge Representation and Reasoning

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# Outline

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# Prisoner's Dilemma

Both prisoners benefit if they cooperate. If one prisoner defects and the other does not, then the defecting prisoner gets out free!

| | cooperate | defect |
|---|---|---|
| cooperate | 2,2 | 0,3 |
| defect | 3,0 | 1,1 |

# Setup

An $n$-player game $(N, A, u)$ consists of

- Set of players $N = \{1, \ldots, n\}$
- $A = A_1 \times \cdots A_n$ where $A_i$ is the action set of player $i$
  - $a \in A$ is an action profile.
  - $u = (u_1, \ldots, u_n)$ specifies a utility function $u_i : A \to \mathbb{R}$ for each player.

# Bimatrix (2-player) Games

| | $a_2^1$ | $a_2^2$ |
|---|---|---|
| $a_1^1$ | $u_1(a_1^1, a_2^1), u_2(a_1^1, a_2^1)$ | $u_1(a_1^1, a_2^2), u_2(a_1^1, a_2^2)$ |
| $a_1^2$ | $u_1(a_1^2, a_2^1), u_2(a_1^2, a_2^1)$ | $u_1(a_1^2, a_2^2), u_2(a_1^2, a_2^2)$ |

- Actions of player $1 = A_1 = \{a_1^1, a_1^2\}$.
- Actions of player $2 = A_2 = \{a_2^1, a_2^2\}$.

# Prisoner's Dilemma

Both prisoners benefit if they cooperate. If one prisoner defects and the other does not, then the defecting prisoner gets out free!

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# Penalty Shootout

Player 1 (Goal-keeper) wants to match; Player 2 (penalty taker) does not want to match.

| | Left | Right |
|---|---|---|
| Left | +1,-1 | -1,+1 |
| Right | -1,+1 | +1,-1 |

# Zero Sum Games

In zero-sum games, there are two players and for all action profiles $a \in A$, $u_1(a) + u_2(a) = 0$.

## Example

| | Left | Right |
|---|---|---|
| Left | +1,-1 | -1,+1 |
| Right | -1,+1 | +1,-1 |

| | Heads | Tails |
|---|---|---|
| Heads | 1 | -1 |
| Tails | -1 | 1 |

# Rock-Paper-Scissors

Both players draw if they have the same action. Otherwise, playing Scissor wins against Paper, playing Paper wins against Rock, and playing Rock wins against Scissors.

| | Rock | Paper | Scissors |
|---|---|---|---|
| Rock | 0 | -1 | 1 |
| Paper | 1 | 0 | -1 |
| Scissors | -1 | 1 | 0 |

# Battle of the Sexes

Player 1 (wife) prefers Ballet over Football. Player 2 (husband) prefers Football over Ballet. Both prefer being together than going alone.

| | Ballet | Football |
|---|---|---|
| Ballet | 2,1 | 0,0 |
| Football | 0,0 | 1,2 |

# Pareto Optimality

One outcome $o'$ Pareto dominates another outcome $o$ if $o'$ all players prefer $o'$ at least as much as $o$ and at least one player strictly prefers $o'$ to $o$.

Each game admits at least one Pareto optimal outcome.

# Outline

# Best Response

Let $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$.

## Definition (Best Response)

$$a'_i \in BR(a_{-i})$$

iff

$$\forall a_i \in A_i, u_i(a'_i, a_{-i}) \geq u_i(a_i, a_{-i})$$

The best response of a player gives the player maximum possible utility.

# Nash Equilibrium

Let $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$.

**Definition (Best Response)**

$a = (a_1, \ldots, a_n)$ is a (pure) **Nash equilibrium** iff

$$\forall i, a_i \in BR(a_{-i}).$$

A Nash equilibrium is an action profile in which each player plays a best response.

# Battle of the Sexes: Pure Nash Equilibria

| | Ballet | Football |
|---|---|---|
| Ballet | 2,1 | 0,0 |
| Football | 0,0 | 1,2 |

What are the pure Nash equilibria of the game?

# Battle of the Sexes: Pure Nash Equilibria

| | Ballet | Football |
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Pure Nash equilibria:

- (Ballet, Ballet)
- (Football, Football)

# Prisoner's Dilemma

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What are the pure Nash equilibria of the game?

# Prisoner's Dilemma

| | cooperate | defect |
|---|---|---|
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- The only Nash equilibrium is (defect, defect).
- The outcome of (defect,defect) is Pareto dominated by the outcome of (cooperate, cooperate).

# Penalty Shootout

| | Left | Right |
|---|---|---|
| Left | 1 | -1 |
| Right | -1 | 1 |

What are the pure Nash equilibria of the game?

# Penalty Shootout

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What are the pure Nash equilibria of the game?

A pure Nash equilibrium may not exist.

# Complexity of a Computing a Pure Nash Equilibrium

Let us assume there are $n$ players and each player has $m$ actions.

- for each of the $m^n$ possible action profiles, check whether some some player out of the $n$ player has a different action among the $m$ actions that gives more utility.
- Total number of steps: $O(m^n mn) = O(m^{n+1}n)$

# Outline

# Playing pure actions may not be a good idea

**Example (Penalty Shootout)**

| | Left | Right |
|---|---|---|
| Left | 1 | -1 |
| Right | -1 | 1 |

# Mixed Strategies

Recall that the possible set of pure actions of each player $i \in N$ is $A_i$.

- A **pure strategy** is one in which exactly one action is played with probability one.
- A **mixed strategy**: more than one action is played with non-zero probability.

The set of strategies for player $i$ is $S_i = \Delta(A_i)$ where $\Delta(A_i)$ is the set of probability distributions over $A_i$.

The set of all strategy profiles is $S = S_1 \times \cdots \times S_n$.

# Mixed Strategies

We want to analyze the payoff of players under a mixed strategy profile:

$$u_i = \sum_{a \in A} u_i(a) Pr(a \mid s)$$

$$Pr(a \mid s) = \prod_{j \in N} s_j(a_j)$$

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## Example (Penalty Shootout)

| | Left | Right |
|---|---|---|
| Left | 1 | -1 |
| Right | -1 | 1 |

Consider the following strategy profile Player $1$ plays Left with probability 0.1 and Right with probability 0.9. Player $2$ players Left with probability 0.1 and Right with probability 0.9.

**Question:** What is the utility of player $1$ under the strategy profile?

# Mixed Strategies

We want to analyze the payoff of players under a mixed strategy profile:

$$u_i = \sum_{a \in A} u_i(a) Pr(a \mid s)$$

$$Pr(a \mid s) = \prod_{j \in N} s_j(a_j)$$

## Example (Penalty Shootout)

| | Left | Right |
|---|---|---|
| Left | 1 | -1 |
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Consider the following strategy profile Player $1$ plays Left with probability 0.1 and Right with probability 0.9. Player $2$ players Left with probability 0.1 and Right with probability 0.9.

Then $u_1 = (0.1 \times 0.1)1 + (0.1 \times 0.9)(-1) + (0.9 \times 0.1)(-1) + (0.9 \times 0.9)(1) = 0.01 - 0.09 - 0.09 + 0.81 = 0.64$ .

# Mixed Strategies

**Definition (Best Response)**

Best response: $s'_i \in BR(s_{-i})$ iff $\forall s_i \in S_i$, $u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i})$.

The best response of a player gives the player maximum possible utility.

**Definition (Nash equilibrium)**

$s = (s_1, \ldots, s_n)$ is a Nash equilibrium iff $\forall i \in N$, $s_i \in BR(s_{-i})$.

A Nash equilibrium is an action profile in which each player plays a best response.

# Nash's Theorem

**Theorem (Nash's Theorem)**

*A mixed Nash equilibrium always exists.*

# Battle of the Sexes

| | Ballet | Football |
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# Battle of the Sexes

| | Ballet | Football |
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- Let us assume that both players play their full support.
- Player 2 plays B with $p$ and F with probability $1 - p$.
- Player 1 must be indifferent between the actions it plays.

$$\begin{aligned} 2(p) + 0(1-p) &= 0p + 1(1-p) \\ p &= 1/3. \end{aligned}$$

- Player 1 plays B with $q$ and F with probability $1 - q$
- Player 2 must be indifferent between the actions it plays.

$$\begin{aligned} 1(q) + 0(1-q) &= 0q + 2(1-q) \\ q &= 1/3. \end{aligned}$$

Thus the mixed strategies $(2/3, 1/3), (1/3, 2/3)$ are in Nash equilibrium.

# Support Enumeration Algorithm

For 2-player games, a support profile can be checked for Nash equilibria as follows:

$$\sum_{a_{-i} \in A_{-i}} s_{-i}(a_{-i}) u_i(a_i, a_{-i}) = U^* \quad \forall i \in N, a_i \in B_i$$

$$\sum_{a_{-i} \in A_{-i}} s_{-i}(a_{-i}) u_i(a_i, a_{-i}) \leq U^* \quad \forall i \in N, a_i \notin B_i$$

$$s_i(a_i) \geq 0 \quad \forall i \in N, a_i \in B_i$$

$$s_i(a_i) = 0 \quad \forall i \in N, a_i \notin B_i$$

$$\sum_{a_i \in A_i} s_i(a_i) = 1$$

When there are more than two players, the constraints are not linear.

# Complexity of Computing Nash Equilibrium

**PPAD** (Polynomial Parity Arguments on Directed graphs) is a complexity class of computational problems for which a solution always exists because of a parity argument on directed graphs.
The class PPAD introduced by Christos Papadimitriou in 1994.
**Representative PPAD problem**: Given an exponential-size directed graph with no isolated nodes and with every node having in-degree and out-degree at most one described by a polynomial-time computable function $f(v)$ that outputs the predecessor and successor of $v$, and a node $s$ with degree 1, find a $t \neq s$ that is either a source or a sink.

**Theorem (Daskalakis et al., Chen & Deng; 2005)**

*The problem of finding a Nash equilibrium is PPAD-complete.*

- It is believed that P is not equivalent to PPAD.
- PPAD-hardness is viewed as evidence that the problem does not admit an efficient algorithm.

# Outline

# Reading

- K. Leyton-Brown and Y. Shoham, Essentials of Game Theory: A Concise Multidisciplinary Introduction. Synthesis Lectures on Artificial Intelligence and Machine Learning. Morgan & Claypool Publishers, 2008. `www.gtessentials.org`
- Y. Shoham and K. Leyton-Brown. Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations. 2009. `http://www.masfoundations.org`